

# Kählerian K3 surfaces and Niemeier lattices, I

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## Abstract

Using results (especially see Remark 1.14.7) of our paper [12], 1979, we clarify relation between Kählerian K3 surfaces and Niemeier lattices. We want to emphasize that all twenty four Niemeier lattices are important for K3 surfaces, not only the one which is related to the Mathieu group.

## 1 Introduction

Studying of finite symplectic automorphism groups of Kählerian K3 surfaces started in our papers [10] (announcement, 1976) and [11] (1979). Some general theory of such groups was developed, and Abelian such groups were classified (14 non-trivial Abelian groups). Further, in [12, Remark 1.14.7] (1979) we showed, in particular, that all finite symplectic automorphism groups of K3 surfaces can be obtained from negative definite even unimodular lattices and their automorphism groups using primitive embeddings of even negative definite lattices into such unimodular lattices (see Sect. 3 here for a review). By our results about existence of primitive embeddings of even lattices into even unimodular lattices in [12], for K3 surfaces it is enough to use negative definite even unimodular lattices of the rank 24. They are Niemeier lattices.

Later, all finite symplectic automorphism groups of Kählerian K3 surface were classified as abstract groups by Mukai [7] (1988), (see also Xiao [19], 1996). Kondō in [5] (1998) showed that this classification can be obtained by using primitive embeddings of lattices into Niemeier lattices (see also the important appendix to this paper by Mukai). This is similar to our considerations in [12, Remark 1.14.7]. Recently, Hashimoto [4] applied similar ideas of using Niemeier lattices to classify finite symplectic automorphism groups of Kählerian K3 surfaces similarly to our results about Abelian such groups in [11].

Thus, now it is clear that the methods of using negative definite even unimodular lattices and Niemeier lattices are very powerful.

In this paper, we use ideas and results of [12, Remark 1.14.7] to show that all twenty four Niemeier lattices are important for Kählerian K3 surfaces, their geometry and their

symplectic automorphism groups. (Usually, the Niemeier lattice with the root system  $24A_1$  related to Mathieu group  $M_{24}$  is used.) From our point of view, all twenty four Niemeier lattices are important for K3 surfaces.

We introduce and use a *marking* of a Kählerian K3 surface  $X$  by a Niemeier lattice. Using this marking, one can study, in particular, finite symplectic automorphism groups and non-singular rational curves on  $X$ . In [11], we demonstrated that to study finite symplectic automorphism groups of K3 surfaces, it is important to work not with algebraic K3 surfaces but with arbitrary Kählerian K3 surfaces. General Kählerian K3 surfaces have negative definite Picard lattices  $S_X$  of the rank  $\text{rk } S_X \leq 19$ . By our results in [12], there exists a primitive embedding of  $S_X$  into one of 24 Niemeier lattices. One can study some arithmetic and geometry of  $S_X$  and of  $X$  using such primitive embedding. It is called a *marking of  $X$* . All 24 Niemeier lattices are important for that. For Kählerian K3 surfaces with semi-negative definite and hyperbolic Picard lattice  $S_X$  we use some modification of these markings.

In Sect. 2, we remind to a reader our results from [12] about primitive embeddings of lattices into even unimodular lattices.

In Sect. 3, we remind to a reader classification of Niemeier lattices and results of [12, Remark 1.14.7] about using of them (or any other even negative definite unimodular lattices) for studying of automorphism groups which act trivially on the discriminant group for negative definite even lattices by applying their primitive embeddings into even unimodular lattices.

In Sect. 4, we remind to a reader definitions and basic results related to Kählerian K3 surfaces  $X$ . Using these results, we introduce markings  $S \subset N_i$  of  $X$  by Niemeier lattices  $N_i$ .

In Sect. 5, we consider applications of markings of K3 surfaces  $X$  by Niemeier lattices to study finite symplectic automorphism groups and non-singular rational curves of  $X$ .

In Sect. 6, we consider examples of markings of Kählerian K3 surfaces by concrete Niemeier lattices  $N_i$ ,  $1 \leq i \leq 18$ , and  $N_{23}$ , and their applications. In particular, we show that for each of Niemeier lattices  $N_i$ ,  $i = 1, 2, 3, 5-9, 11-18, 23$ , there exists a Kählerian K3 surface  $X$  such that  $X$  can be marked by the Niemeier lattice  $N_i$  only. See Theorem 13. We believe that similar results are valid for all remaining Niemeier lattices.

We hope to consider other examples and applications in further publications.

## 2 Existence of a primitive embedding of an even lattice into even unimodular lattices, according to [12]

In this paper, we use notations, definitions and results of [12] about lattices (that is non-degenerate integral (over  $\mathbb{Z}$ ) symmetric bilinear forms). In particular,  $\oplus$  denotes the orthogonal sum of lattices, quadratic forms. For a prime  $p$ , we denote by  $\mathbb{Z}_p$  the ring of  $p$ -adic integers, and by  $\mathbb{Z}_p^*$  its group of invertible elements.

Let  $S$  be a lattice. Let  $A_S = S^*/S$  be its discriminant group, and  $q_S$  its discriminant quadratic form on  $A_S$  where we assume that the lattice  $S$  is even: that is  $x^2$  is even for any  $x \in S$ . We denote by  $l(A_S)$  the minimal number of generators of the finite Abelian group  $A_S$ , and by  $|A_S|$  its order. For a prime  $p$ , we denote by  $q_{S_p} = q_{S \otimes \mathbb{Z}_p}$  the  $p$ -component of  $q_S$  (equivalently, the discriminant quadratic form of the  $p$ -adic lattice  $S \otimes \mathbb{Z}_p$ ). A quadratic form on a group of order 2 is denoted by  $q_\theta^{(2)}(2)$ . A  $p$ -adic lattice  $K(q_{S_p})$  or the rank  $l(A_{S_p})$  with the discriminant quadratic form  $q_{S_p}$  is denoted by  $K(q_{S_p})$ . It is unique, up to isomorphisms, for  $p \neq 2$ , and for  $p = 2$ , if  $q_{S_2} \not\cong q_\theta^{(2)}(2) \oplus q'$ . We have the following result where an embedding  $S \subset L$  of lattices is called *primitive* if  $L/S$  has no torsion.

**Theorem 1.** (*Theorem 1.12.2 in [12]*).

*Let  $S$  be an even lattice of the signature  $(t_{(+)}, t_{(-)})$ , and  $l_{(+)}, l_{(-)}$  are integers.*

*Then, there exists a primitive embedding of  $S$  into one of even unimodular lattices of the signature  $(l_{(+)}, l_{(-)})$  if and only if the following conditions satisfy:*

- (1)  $l_{(+)} - l_{(-)} \equiv 0 \pmod{8}$ ;
- (2)  $l_{(+)} - t_{(+)} \geq 0$ ,  $l_{(-)} - t_{(-)} \geq 0$ ,  $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} \geq l(A_S)$ ;
- (3)  $(-1)^{l_{(+)} - t_{(+)}} |A_S| \equiv \det K(q_{S_p}) \pmod{(\mathbb{Z}_p^*)^2}$  for each odd prime  $p$  such that  $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(A_{S_p})$ ;
- (4)  $|A_S| \equiv \pm \det K(q_{S_2}) \pmod{(\mathbb{Z}_2^*)^2}$ , if  $l_{(+)} + l_{(-)} - t_{(+)} - t_{(-)} = l(A_{S_2})$  and  $q_{S_2} \not\cong q_\theta^{(2)}(2) \oplus q'$ .

Remark that if the last inequality in (2) is strict then one does not need the conditions (3) and (4). If  $q_{S_2} \cong q_\theta^{(2)}(2) \oplus q'$ , then one does not need the condition (4).

## 3 Niemeier lattices and their primitive sublattices

Further, negative definite even unimodular lattices  $N$  of the rank 24 are called *Niemeier lattices*. They were classified by Niemeier [8]. See also [3, Ch. 16, 18]. All elements with square  $(-2)$  of these lattices define root systems  $\Delta(N)$  and generate root sublattices  $[\Delta(N)] = N^{(2)} \subset N$  which are orthogonal sums of the root lattices  $A_n$ ,  $n \geq 1$ ,  $D_m$ ,  $m \geq 4$ ,

or  $E_k$ ,  $k = 6, 7, 8$  with the corresponding root systems  $A_n, D_m, E_k$ . We denote by the same letters their Dynkin diagrams. We fix standard bases by simple roots of these root systems and lattices according to [1] (see Figure 1). If there are several components, we put an additional second index which numerates components.

By Niemeier [8], there are 24 Niemeier lattices  $N$ , up to isomorphisms, and they are characterized by their root sublattices. Here is the list where  $\oplus$  denotes the orthogonal sum of lattices:

$$N^{(2)} = [\Delta(N)] =$$

$$(1) D_{24}, (2) D_{16} \oplus E_8, (3) 3E_8, (4) A_{24}, (5) 2D_{12}, (6) A_{17} \oplus E_7, (7) D_{10} \oplus 2E_7,$$

$$(8) A_{15} \oplus D_9, (9) 3D_8, (10) 2A_{12}, (11) A_{11} \oplus D_7 \oplus E_6, (12) 4E_6, (13) 2A_9 \oplus D_6,$$

$$(14) 4D_6, (15) 3A_8, (16) 2A_7 \oplus 2D_5, (17) 4A_6, (18) 4A_5 \oplus D_4, (19) 6D_4,$$

$$(20) 6A_4, (21) 8A_3, (22) 12A_2, (23) 24A_1$$

give 23 Niemeier lattices  $N_i$ , where the number  $i$  is shown in brackets above. The last one, the *Leech lattice* (24) with  $N^{(2)} = \{0\}$  has no roots. Further, we also denote by  $N(R)$  the Niemeier lattice with the root system  $R$ .

We recall that a basis  $P(N)$  of the root lattice  $[\Delta(N)]$  by simple roots is defined up to the reflection group  $W(N)$  which is generated by reflections  $s_\delta : x \rightarrow x + (x \cdot \delta)\delta$ ,  $x \in N$ , in roots  $\delta \in \Delta(N)$ . We denote by  $\Gamma(P(N))$  its Dynkin diagram. Let  $A(N) \subset O(N)$  be the group of symmetries of the fixed basis  $P(N)$ . Thus,  $g \in A(N)$  if and only if  $g(P(N)) = P(N)$ . Then we have the semi-direct product

$$O(N) = A(N) \ltimes W(N).$$

for the automorphism group of  $N$ . Equivalently,  $P(N)$  is equivalent to a choice of the fundamental chamber for  $W(N)$ , and  $A(N)$  is the group of symmetries of the fundamental chamber.

We denote by  $\mathcal{N}$  the disjoint union of all 24 Niemeier lattices  $N_i$ ,  $i = 1, 2, \dots, 24$  with their zeros identified. Thus, naturally,  $\mathcal{N}_i = N_i \subset \mathcal{N}$  denotes the corresponding Niemeier sublattice of this set which is called a *Niemeier component* of  $\mathcal{N}$ . If  $K$  is a lattice, then an embedding  $K \subset \mathcal{N}$  means an embedding of  $K$  to one of  $\mathcal{N}_i \subset \mathcal{N}$  as a sublattice. If  $K \neq \{0\}$ , then  $\mathcal{N}_i$  is defined uniquely, and it is called the *component* of the embedding  $K \subset \mathcal{N}$ . The embedding is called *primitive* if  $\mathcal{N}_i/K$  has no torsion.

Further, we fix bases  $P(\mathcal{N}_i)$  for  $\Delta(\mathcal{N}_i)$  for all 24 components of  $\mathcal{N}$ . Naturally, the direct product

$$A(\mathcal{N}) = \prod_{i=1}^{i=24} A(\mathcal{N}_i)$$

acts on  $\mathcal{N}$ . By definition,  $A(\mathcal{N}_i)$  acts as  $A(N_i)$  on the component  $\mathcal{N}_i = N_i$ , and  $A(\mathcal{N}_i)$  is identity on all other components  $\mathcal{N}_j$ ,  $j \neq i$ .

Since Niemeier lattices have the signature  $(0, 24)$ , from Theorem 1 we obtain

**Theorem 2.** *(Corollary of Theorem 1.12.2 in [12]). An even lattice  $S$  has a primitive embedding into  $\mathcal{N}$  (equivalently, to one of Niemeier lattices) if and only if the following conditions satisfy:*

- (2)  $S$  is negative definite and  $\text{rk } S + l(A_S) \leq 24$ ;
- (3)  $|A_S| \equiv \det K(q_{S_p}) \pmod{(\mathbb{Z}_p^*)^2}$  for each odd prime  $p$  such that  $\text{rk } S + l(A_{S_p}) = 24$ ;
- (4)  $|A_S| \equiv \pm \det K(q_{S_2}) \pmod{(\mathbb{Z}_2^*)^2}$ , if  $\text{rk } S + l(A_{S_2}) = 24$  and  $q_{S_2} \not\cong q_\theta^{(2)}(2) \oplus q'$ .

Theorem 2 gives simple conditions on  $S$  when it can be considered as a primitive sublattice of one of Niemeier lattices (equivalently, of  $\mathcal{N}$ ). We can use it to get an important information about  $S$  and all primitive sublattices of Niemeier lattices. Later, we shall apply these results to Picard lattices of Kählerian K3 surfaces.

For a negative definite lattice  $S$ , we denote by  $H(S)$  the kernel of the natural homomorphism

$$\pi : O(S) \rightarrow O(q_S).$$

Equivalently, an automorphism  $\phi \in O(S)$  of the lattice  $S$  belongs to  $H(S)$  if and only if  $\phi$  gives identity on the discriminant group  $S^*/S$ .

It was observed in [12, Remark 1.14.7] that  $\pi$  has a non-trivial kernel if and only if  $S$  contains some exceptional sublattices. These exceptional sublattices can be found from negative definite even unimodular lattices and their automorphism groups. Remark that for an unimodular lattice  $L$ , the group  $H(L) = O(L)$ . To study  $H(S)$  for negative definite even lattices  $S$ , it was suggested in [12, Remark 1.14.7] to use primitive embeddings of  $S$  into negative definite even unimodular lattices. For example, for lattices  $S$  satisfying Theorem 2, one can use primitive embeddings into Niemeier lattices (equivalently, into  $\mathcal{N}$ ). Below, we summarize these results of [12, Remark 1.14.7].

Let  $S$  be an even negative definite lattice. Let  $\Delta(S)$  be the set of roots  $\delta \in S$  with square  $\delta^2 = -2$ . It is easy to see that the reflection  $s_\delta$  belongs to  $H(S)$ . Thus,  $[\delta] = \langle -2 \rangle$  is the first example of an exceptional sublattice. The Weyl group  $W(S)$  generated by reflections in all elements of  $\Delta(S)$  is the normal subgroup of  $H(S)$ . Let us choose a basis  $P(S)$  of the root system  $\Delta(S)$  (equivalently, a fundamental chamber of the Weyl group  $W(S)$ ). Let

$$A(S) = \{\phi \in H(S) \mid \phi(P(S)) = P(S)\}$$

(equivalently,  $A(S) \subset H(S)$  is the symmetry subgroup of the fundamental chamber). Then  $H(S) = A(S) \ltimes W(S)$  is the semi-direct product. Let

$$\mathcal{L}(S) = S_{A(S)} = (S^{A(S)})^\perp_S$$

be the orthogonal complement to the fixed part  $S^{A(S)}$  of the action of  $A(S)$  in  $S$ . Obviously, the sublattice  $\mathcal{L}(S) = S_{A(S)} \subset S$  is defined uniquely up to the action of  $W(S)$ . It is called *the coinvariant sublattice of  $S$  for  $A(S)$* .

It was shown in Proposition 1.14.8 of [12, Remark 1.14.7] that the primitive sublattice  $\mathcal{L}(S)$  satisfies the following properties:  $\mathcal{L}(S)^{A(S)} = \{0\}$  (this is obvious),  $\mathcal{L}(S)$  has no roots with square  $(-2)$ ,  $\mathcal{L}(\mathcal{L}(S)) = \mathcal{L}(S)$ , and  $A(\mathcal{L}(S)) = A(S)|_{\mathcal{L}(S)}$ . Thus,  $H(\mathcal{L}(S)) = A(\mathcal{L}(S))$ ,  $A(\mathcal{L}(S))$  defines  $A(S)$ : one should continue  $A(\mathcal{L}(S))$  identically to the orthogonal complement of  $\mathcal{L}(S)$  in  $S$ .

The sublattice  $\mathcal{L}(S) = S_{A(S)}$  was called in [12, Remark 1.14.7] as *the Leech type sublattice of  $S$* . Thus, the group  $H(S)$  and its natural subgroups  $W(S)$  and  $A(S)$  are completely defined by the basis  $P(S)$  and the Leech type sublattice  $\mathcal{L}(S)$ . Really,  $W(S)$  is generated by reflections in  $P(S)$ , and  $A(S) = A(\mathcal{L}(S))$  if one continues  $A(\mathcal{L}(S))$  identically to the orthogonal complement  $\mathcal{L}(S)^\perp_S$ .

Now let us assume that  $S \subset N_i$  is a primitive sublattice of one of Niemeier lattices (or of any other even unimodular lattice). Then, replacing  $S$  by  $w(S)$  where  $w \in W(N_i)$ , one can assume that  $P(S)$  is a subset of  $P(N_i)$ . Later, we always assume that the primitive embedding  $S \subset N_i$  satisfies this condition:

$$P(S) = S \cap P(N_i) \text{ is a basis of } \Delta(S). \quad (1)$$

Then, continuing  $A(S)$  identically to  $S^\perp_{N_i}$  (it is possible because  $A(S)$  gives identity on  $S^*/S$ ), one obtains a subgroup of  $A(N_i)$ . It follows that

$$A(S) = \{\phi|_S \mid \phi \in A(N_i) \text{ and } \phi|_{S^\perp_{N_i}} \text{ is identity}\}. \quad (2)$$

It follows that  $A(S) \subset A(N_i)$  and  $P(S) \subset P(N_i)$  satisfy the obvious important condition:

$$P(S) \text{ is invariant with respect to } A(S).$$

Since for Niemeier lattices  $N_i$  the sets  $P(N_i)$  and their graphs  $\Gamma(P(N_i))$ , and the groups  $A(N_i)$  are known, we obtain an effective tool to construct primitive sublattices  $S \subset N_i$  with different graphs  $\Gamma(P(S))$  and groups  $A(S)$ .

**Example.** The Niemeier lattices  $N = N_1 = N(D_{24})$ ,  $N_2 = N(D_{16} \oplus E_8)$  have the trivial group  $A(N)$  (see [3, Ch. 16]). It follows that any primitive sublattice  $S \subset N$  of such lattices has the trivial group  $A(S)$  and the trivial Leech type sublattice  $\mathcal{L}(S)$ .

Let us assume that graph  $\Gamma(S^{(2)})$  has a subgraph  $\mathbb{D}_n$  where  $n \geq 13$ . By the classification of Niemeier lattices, then  $S$  can have a primitive embedding only to the lattices  $N(D_{24})$  and  $N(D_{16} \oplus E_8)$ , and the group  $A(S)$  is trivial again if  $S$  satisfies Theorem 2.

We shall consider other examples in section 5.

## 4 Additional markings of Kählerian K3 surfaces by Niemeier lattices

We consider Kählerian K3 surfaces. We recall that they are Kählerian compact complex surfaces  $X$  such that  $X$  is simply-connected and  $X$  has a holomorphic 2-dimensional differential form  $\omega_X \in H^{2,0}(X)$  such that  $\omega_X$  has no zeros. One can consider  $\omega_X$  as a complex volume form. See [15, Chapter 9] and [2] about Kählerian K3 surfaces.

It is known that the cohomology lattice  $H^2(X, \mathbb{Z})$  with the intersection pairing is an even unimodular lattice of the signature  $(3, 19)$ . It has no torsion. All even unimodular lattices of the signature  $(3, 19)$  are isomorphic (the same is valid for all indefinite even unimodular lattices of the same signature).  $X$  has the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(X) + H^{1,1}(X) + H^{0,2}(X)$$

where  $H^{2,0}(X) = \mathbb{C}\omega_X$  is 1-dimensional,  $H^{0,2}(X) = \overline{H^{2,0}(X)}$  and  $H^{1,1}(X) = \overline{H^{1,1}(X)}$  is 20-dimensional. It follows that the *Picard lattice*

$$S_X = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

of  $X$  is a sublattice of the hyperbolic real subspace  $H_{\mathbb{R}}^{1,1}(X) \subset H^{1,1}(X)$  of the signature  $(1, 19)$ . It follows that the Picard lattice  $S_X$  is a primitive sublattice of  $H^2(X, \mathbb{Z})$ , and it satisfies to one of the following conditions:

- (a)  $S_X$  is negative definite of the rank  $0 \leq \text{rk } S_X \leq 19$ ;
- (b)  $S_X$  is semi-negative definite with 1-dimensional kernel, and  $1 \leq \text{rk } S_X \leq 19$ ;
- (c)  $S_X$  is hyperbolic (that is it has the signature  $(1, \text{rk } S_X - 1)$ ), and  $1 \leq \text{rk } S_X \leq 20$  (this is the case when  $X$  is algebraic).

Further, we denote by  $\rho(X) = \text{rk } S_X$  the Picard number of  $X$ .

Let  $L_{K3}$  be an abstract even unimodular lattice of the signature  $(3, 19)$ . Thus, the Picard lattice  $S_X$  of a Kählerian K3 surface  $X$  has a primitive embedding  $S_X \subset L_{K3}$ , and it satisfies to one of conditions (a), (b) or (c). By the epimorphicity of the period map for K3 surfaces [6], [16], [18], Picard lattices  $S_X$  of K3 surfaces are characterized by these properties. An

even lattice which is either negative definite, or semi-negative definite with one-dimensional kernel, or hyperbolic is isomorphic to the Picard lattice of one of Kählerian K3 surfaces if and only if it has a primitive embedding into the lattice  $L_{K3}$ .

By Theorem 1 (from [12]), we obtain the complete description of Picard lattices of K3 surfaces.

**Theorem 3.** (Corollary of Theorem 1.12.2 in [12]).

An even negative definite lattice  $M$  is isomorphic to the Picard lattice  $S_X$  of one of Kählerian K3 surfaces  $X$  (equivalently,  $M$  has a primitive embedding into  $L_{K3}$ ) if and only if

- (2)  $\text{rk } M \leq 19$  and  $\text{rk } M + l(A_M) \leq 22$ ;
- (3)  $|A_M| \equiv \det K(q_{M_p}) \pmod{(\mathbb{Z}_p^*)^2}$  for each odd prime  $p$  such that  $\text{rk } M + l(A_{M_p}) = 22$ ;
- (4)  $|A_M| \equiv \pm \det K(q_{M_2}) \pmod{(\mathbb{Z}_2^*)^2}$ , if  $\text{rk } M + l(A_{M_2}) = 22$  and  $q_{M_2} \not\cong q_\theta^{(2)}(2) \oplus q'$ .

**Theorem 4.** (Corollary of Theorem 1.12.2 in [12]).

An even semi-negative definite lattice  $M$  with one-dimensional kernel  $\text{Ker } M$  is isomorphic to the Picard lattice  $S_X$  of one of Kählerian K3 surfaces  $X$  (equivalently,  $M$  has a primitive embedding into  $L_{K3}$ ), if and only if for  $\widetilde{M} = M/\text{Ker } M$  one has

- (2)  $\text{rk } \widetilde{M} \leq 18$  and  $\text{rk } \widetilde{M} + l(A_{\widetilde{M}}) \leq 20$ ;
- (3)  $|A_{\widetilde{M}}| \equiv \det K(q_{\widetilde{M}_p}) \pmod{(\mathbb{Z}_p^*)^2}$  for each odd prime  $p$  such that  $\text{rk } \widetilde{M} + l(A_{\widetilde{M}_p}) = 20$ ;
- (4)  $|A_{\widetilde{M}}| \equiv \pm \det K(q_{\widetilde{M}_2}) \pmod{(\mathbb{Z}_2^*)^2}$ , if  $\text{rk } \widetilde{M} + l(A_{\widetilde{M}_2}) = 20$  and  $q_{\widetilde{M}_2} \not\cong q_\theta^{(2)}(2) \oplus q'$ .

**Theorem 5.** (Corollary of Theorem 1.12.2 in [12]).

An even hyperbolic lattice  $M$  (that is  $M$  has the signature  $(1, \text{rk } M - 1)$ ) is isomorphic to the Picard lattice  $S_X$  of one of algebraic K3 surfaces  $X$  over  $\mathbb{C}$  (equivalently,  $M$  has a primitive embedding into  $L_{K3}$ ) if and only if

- (2)  $\text{rk } M \leq 20$  and  $\text{rk } M + l(A_M) \leq 22$ ;
- (3)  $|A_M| \equiv \det K(q_{M_p}) \pmod{(\mathbb{Z}_p^*)^2}$  for each odd prime  $p$  such that  $\text{rk } M + l(A_{M_p}) = 22$ ;
- (4)  $|A_M| \equiv \pm \det K(q_{M_2}) \pmod{(\mathbb{Z}_2^*)^2}$ , if  $\text{rk } M + l(A_{M_2}) = 22$  and  $q_{M_2} \not\cong q_\theta^{(2)}(2) \oplus q'$ .



We recall the definition of the period domain  $\tilde{\Omega}$  for Kählerian K3 surfaces (see [2]). We fix an even unimodular lattice  $L_{K3}$  of the signature  $(3, 19)$ . The  $\tilde{\Omega}$  consists of all triplets

$$(H^{2,0}, V^+, P) \quad (3)$$

which we describe below.

Here  $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$  is a one-dimensional complex linear subspace satisfying the condition

$$\omega \cdot \omega = 0, \quad \omega \cdot \bar{\omega} > 0 \quad (4)$$

for any  $0 \neq \omega \in H^{2,0}$ . (We extend the symmetric bilinear form of  $L_{K3}$  to the symmetric  $\mathbb{C}$ -bilinear form on  $L_{K3} \otimes \mathbb{C}$ .) Such  $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$  define a 20-dimensional complex homogeneous manifold which is denoted by  $\Omega$ .

For a fixed  $H^{2,0} \subset L_{K3} \otimes \mathbb{C}$  from  $\Omega$ , we denote by  $H_{\mathbb{R}}^{1,1}$  the orthogonal complement to  $H^{2,0}$  in  $L_{K3} \otimes \mathbb{R}$  which is a hyperbolic form of the signature  $(1, 19)$ . It contains the light cone

$$V = \{x \in H_{\mathbb{R}}^{1,1} \mid x^2 > 0\}. \quad (5)$$

The  $V^+$  denotes one of two halves (that is connected components) of this cone  $V$ . It defines the hyperbolic space  $\mathcal{H}^+ = V^+/\mathbb{R}^+$  which is the projectivization of the half-cone  $V^+$ .

Let  $H_{\mathbb{Z}}^{1,1} = H_{\mathbb{R}}^{1,1} \cap L_{K3}$ . The  $H_{\mathbb{Z}}^{1,1}$  is a lattice which is either negative definite, or semi-negative definite with one-dimensional kernel, or hyperbolic. Let  $\Delta(H_{\mathbb{Z}}^{1,1})$  be the set of all elements with square  $(-2)$  of this lattice. The group  $W(H_{\mathbb{Z}}^{1,1})$  generated by reflections in all elements from  $\Delta(H_{\mathbb{Z}}^{1,1})$  is the discrete reflection group in  $V^+$  and  $\mathcal{H}^+$ . The set  $P$  is the set of perpendicular vectors from  $\Delta(H_{\mathbb{Z}}^{1,1})$  to one of fundamental chambers  $\mathcal{M}$  of this reflection group which are directed outwards of this chambers. Thus

$$\mathcal{M} = \{x \in V^+ \mid x \cdot P \geq 0\}, \quad (6)$$

and each codimension one face of  $\mathcal{M}$  is perpendicular to exactly one  $\delta \in P$ , and vice versa each  $\delta \in P$  is perpendicular to a codimension one face of  $\mathcal{M}$ . Shortly,  $P = P(\mathcal{M})$ .

The set of the triplets (3) is denoted by  $\tilde{\Omega}$ . It is a non-Hausdorff 20-dimensional complex manifold which gives an étale covering of  $\Omega$ .

We recall that to each Kählerian K3 surface  $X$  one can correspond the canonical triplet

$$(H^{2,0}(X), V^+(X), P(X)). \quad (7)$$

See [2] for details.

Here  $H^{2,0}(X) \subset H^2(X, \mathbb{Z}) \otimes \mathbb{C}$  was introduced and considered above.

Here  $V^+(X)$  is the half cone containing the Kähler class of the light cone

$$V(X) = \{x \in H_{\mathbb{R}}^{1,1}(X) \mid x^2 > 0\}. \quad (8)$$

Here

$$P(X) \subset S_X \quad (9)$$

is the set of classes of all non-singular rational curves on  $X$ . All of them have the square  $(-2)$ . Any exceptional curve on  $X$  (that is an irreducible complex curve  $C \subset X$  with  $C^2 < 0$ ) is one of them. They define the nef cone

$$NEF(X) = \{x \in \overline{V^+(X)} \mid x \cdot P(X) \geq 0\}$$

which is the fundamental chamber for the reflection group  $W(S_X)$ , and  $P(X)$  is the set of perpendicular vectors to codimension one faces of  $NEF(X)$ .

We recall that a marking of a Kählerian K3 surface  $X$  is an isomorphism

$$\alpha : H^2(X, \mathbb{Z}) \cong L_{K3} \quad (10)$$

of lattices where the lattice  $L_{K3}$  was introduced above. A pair  $(X, \alpha)$  is called a *marked Kählerian K3 surface*.

By taking

$$\begin{aligned} & \alpha(H^{2,0}(X), V^+(X), P(X)) = \\ & ((\alpha \otimes \mathbb{C})(H^{2,0}(X)), (\alpha \otimes \mathbb{R})(V^+(X)), \alpha(P(X))) \in \tilde{\Omega} \end{aligned} \quad (11)$$

we obtain the *period map*  $\alpha$  from the moduli space of marked Kählerian K3 surfaces to the period domain  $\tilde{\Omega}$  of marked Kählerian K3 surface.

By the *Global Torelli Theorem for Kählerian K3 surfaces* (see [14] (for algebraic case) and [2]), and the *epimorphicity of the period map for Kählerian K3 surfaces* (see [6] (for algebraic case), and [16], [18]), the period map  $\alpha$  is the isomorphism of the complex spaces.

We want to introduce an additional marking of Kählerian K3 surfaces by Niemeier lattices  $N_i$  (or by  $\mathcal{N}$ ).

We introduce the corresponding period domain  $\tilde{\Omega}_{\mathcal{N}}$  which consists of all quadruples

$$(H^{2,0}, V^+, P, \tau : S \subset N_i) \quad (12)$$

where the triplet  $(H^{2,0}, V^+, P) \in \tilde{\Omega}$  is as above.

Now, we shall describe  $\tau : S \subset N_i$ . Here  $S \subset H_{\mathbb{Z}}^{1,1}$  is the maximal negative definite sublattice of  $H_{\mathbb{Z}}^{1,1}$ . Thus,  $S = H_{\mathbb{Z}}^{1,1}$  if  $H_{\mathbb{Z}}^{1,1}$  is negative definite;  $H_{\mathbb{Z}}^{1,1} = S \oplus \mathbb{Z}c$  if  $H_{\mathbb{Z}}^{1,1}$  has a one-dimensional kernel  $\mathbb{Z}c$ ; and  $S \subset H_{\mathbb{Z}}^{1,1}$  is a primitive negative definite sublattice of  $H_{\mathbb{Z}}^{1,1}$ .

such that  $(S)_{H_{\mathbb{Z}}^{1,1}}^{\perp} = \mathbb{Z}h$  where  $h^2 > 0$ , if  $H_{\mathbb{Z}}^{1,1}$  is hyperbolic. For the parabolic case, when  $S$  has a one-dimensional kernel  $\mathbb{Z}c$ , we additionally require that  $P \cap S = P(S)$ . For the hyperbolic case, we additionally require that  $h$  is *nef*, that is  $h \in V^+$  and  $h \cdot P \geq 0$ . Equivalently,  $h$  belongs to the fundamental chamber for  $W(H_{\mathbb{R}}^{1,1})$  defined by the  $V^+$  and  $P$ . The  $\tau : S \subset N_i$  is a primitive embedding of  $S$  into one of Niemeier lattices  $N_i$  (equivalently, to  $\mathcal{N}$ ) such that  $\tau(P \cap S) \subset P(N_i)$ . Thus,  $\tau$  can be considered as an additional marking by Niemeier lattices. Since  $S$  is negative definite and has a primitive embedding to  $L_{K3}$ , it satisfies the Theorem 3. In particular,  $\text{rk } S \leq 19$  and  $\text{rk } S + l(A_S) \leq 22$ . Then  $\text{rk } S + l(A_S) < 24$ . Thus,  $S$  satisfies Theorem 2 and  $S$  has a primitive embedding to one of 24 Niemeier lattices  $N_i$ . The same proof shows that  $S \oplus A_1$  satisfies Theorem 2. It follows that  $S$  has a primitive embedding into one of 23 Niemeier lattices with non-empty set of roots. *This is the important trick due to Kondō [5] which permits to avoid the difficult Leech lattice.*

It follows that

$$\tau : \tilde{\Omega}_{\mathcal{N}} \rightarrow \tilde{\Omega}$$

is a natural “covering” of  $\tilde{\Omega}$  which is onto. Unfortunately, we cannot claim that  $\tilde{\Omega}_{\mathcal{N}}$  is a manifold. We can only claim that it is a topological space with open subsets which are pre-images of open subsets from  $\tilde{\Omega}$ . We can only claim that  $\tau$  is one to one over general points of  $\tilde{\Omega}$  where  $H_{\mathbb{Z}}^{1,1} = \{0\}$  since we identify in  $\mathcal{N}$  zeros of all Niemeier lattices  $N_i$ ,  $i = 1, 2, \dots, 24$ . This points of  $\tilde{\Omega}$  give a complement to infinite number of divisors of  $\tilde{\Omega}$ . Thus, we can only claim that  $\tilde{\Omega}_{\mathcal{N}}$  is similar to a manifold over these points of  $\tilde{\Omega}$ . Moreover,  $\tau$  has finite fibres over points of  $\tilde{\Omega}$  where  $H_{\mathbb{Z}}^{1,1}$  is negative definite. Thus,  $\tilde{\Omega}_{\mathcal{N}}$  is very similar to a manifold over these points. Over other points of  $\tilde{\Omega}$  the map  $\tau$  has countable fibres, in general.

Similarly, we introduce additional marking for Kählerian K3 surfaces  $X$ . It is

$$\tau : S \subset \mathcal{N}. \tag{13}$$

Here  $S \subset S_X$  is a maximal negative definite sublattice. That is  $S = S_X$  if  $S_X$  is negative definite.  $S_X = S \oplus \mathbb{Z}c$  if  $S_X$  is semi-negative definite with one-dimensional kernel generated by the class  $c$  of an elliptic curve  $C$  on  $X$ , and  $S \cap P(X) = P(S)$ . If  $X$  is algebraic, then  $S = h_{S_X}^{\perp}$  where  $h \in S_X$  is primitive,  $h^2 > 0$ , and  $h$  is *nef*, that is  $h \cdot D \geq 0$  for every effective divisor  $D$  on  $X$ . Here  $\tau : S \subset \mathcal{N}$  is a primitive embedding of  $S$  into one of 24 Niemeier lattices  $N_i$  such that  $P(X) \cap S = P(S) \subset P(N_i)$ .

The standard marking  $\alpha : H^2(X, \mathbb{Z}) \cong L_{K3}$  of a Kählerian K3 surface  $X$  with additional marking  $\tau : S \subset \mathcal{N}$  by Niemeier lattices gives its periods  $\alpha(X, \tau) \in \tilde{\Omega}_{\mathcal{M}}$ . They are

$$\alpha(H^{2,0}(X), V^+(X), P(X), \tau : S \subset \mathcal{N}) =$$

$$((\alpha \otimes \mathbb{C})(H^{2,0}(X)), (\alpha \otimes \mathbb{R})(V^+(X)), \alpha(P(X)), \tau\alpha^{-1} : \alpha(S) \subset \mathcal{N}) \in \tilde{\Omega}_{\mathcal{N}}. \quad (14)$$

By the Global Torelli Theorem and the Epimorphicity of Torelli map for Kählerian K3 surfaces (see references above), we obtain isomorphism of moduli spaces of marked Kählerian K3 surfaces  $X$  with additional marking by Niemeier lattices and their periods space  $\tilde{\Omega}_{\mathcal{N}}$ .

**Remark 6.** By these definitions and considerations, to construct Kählerian K3 surfaces  $X$  with a marking  $\tau : S \subset N_i$  by a Niemeier lattice  $N_i$ , one has to follow the following procedure.

(a) Check that  $S$  satisfies Theorem 3 (equivalently, there exists a primitive embedding  $S \subset L_{K3}$ ).

(b) Choose a primitive embedding  $S \subset L_{K3}$ .

(c1) To construct  $X$  with negative definite  $S_X$ , choose  $H^{2,0} \subset (S)_{L_{K3}}^\perp \otimes \mathbb{C}$  which is general enough to have  $H_{\mathbb{Z}}^{1,1} = S$ . Choose  $V^+ \subset H_{\mathbb{R}}^{1,1}$ , and take  $P = P(S) = P(N_i) \cap S$ . Then  $(H^{2,0}, V^+, P, \tau : S \subset N_i)$  gives periods of a marked Kählerian K3 surface  $(X, \alpha)$  with  $S_X = \alpha^{-1}(S)$ , and marking  $\tau\alpha : S_X = \alpha^{-1}(S) \subset N_i$  by the Niemeier lattice  $N_i$ . The set  $P(X) = \alpha^{-1}(P(N_i) \cap S)$ .

(c2) To construct  $X$  with semi-negative definite  $S_X$ , check that  $S \subset L_{K3}$  satisfies Theorem 4. Then there exists a primitive non-zero isotropic  $c \in (S)_{L_{K3}}^\perp$  such that  $S \oplus \mathbb{Z}c \subset L_{K3}$  is a primitive sublattice with the kernel  $\mathbb{Z}c$ . Choose  $H^{2,0} \subset (S \oplus \mathbb{Z}c)_{L_{K3}}^\perp \otimes \mathbb{C}$  which is general enough to have  $H_{\mathbb{Z}}^{1,1} = S \oplus \mathbb{Z}c$ . Choose  $V^+ \subset H_{\mathbb{R}}^{1,1}$  such that  $c \in \overline{V^+}$ . Take  $P = P(S) = P(N_i) \cap S$ , and for each connected component  $P_i, i = 1, 2, \dots, k$ , of the Dynkin diagram of  $P$ , take the maximal root  $\delta_i$ , and denote  $p_i = c - \delta_i$ . Then

$$\tilde{P} = P \cup \{p_1, p_2, \dots, p_k\} = P(S \oplus \mathbb{Z}c),$$

and  $(H^{2,0}, V^+, \tilde{P}, \tau : S \subset N_i)$  gives periods of a marked Kählerian K3 surface  $(X, \alpha)$  with  $S_X = \alpha^{-1}(S \oplus \mathbb{Z}c)$  and with marking  $\tau\alpha : \alpha^{-1}(S) \subset N_i$  by the Niemeier lattice  $N_i$ . The set  $P(X) = \alpha^{-1}(\tilde{P})$ .

(c3) To construct  $X$  with hyperbolic  $S_X$  (thus,  $X$  is algebraic), take a primitive  $h \in (S)_{L_{K3}}^\perp$  such that  $h^2 > 0$  (they exist obviously). Then the primitive sublattice  $\tilde{S} = [S \oplus \mathbb{Z}h]_{pr}$  of  $L_{K3}$  generated by  $S$  and  $h$  is hyperbolic. Choose  $H^{2,0} \subset (\tilde{S})_{L_{K3}}^\perp \otimes \mathbb{C}$  which is general enough to have  $H_{\mathbb{Z}}^{1,1} = \tilde{S}$ . Choose  $V^+ \subset H_{\mathbb{R}}^{1,1}$  such that  $h \in V^+$ . Take  $P = P(S) = P(N_i) \cap S$ , and take the fundamental chamber  $\mathcal{M}$  for  $W(\tilde{S})$  such that  $h \in \mathcal{M}$  and  $P \subset P(\mathcal{M})$  where  $P(\mathcal{M})$  is the set of perpendicular vectors with square  $(-2)$  to codimension one faces of  $\mathcal{M}$  directed outwards of  $\mathcal{M}$ . Then  $(H^{2,0}, V^+, P(\mathcal{M}), \tau : S \subset N_i)$  gives periods of a marked algebraic K3 surface  $(X, \alpha)$  with  $S_X = \alpha^{-1}(\tilde{S})$ ,  $nef$  element  $\alpha^{-1}(h)$ , and  $P(X) = \alpha^{-1}(P(\mathcal{M}))$ , and with the marking  $\tau\alpha : \alpha^{-1}(S) \subset N_i$  by the Niemeier lattice  $N_i$ . We have  $P(X) = \alpha^{-1}(P(\mathcal{M}))$ ,

and the linear system  $|nh|$ ,  $n > 0$ , contracts non-singular rational curves which have classes from  $\alpha^{-1}(S \cap P(N_i))$ , and only these curves.

## 5 Applications of marking by Niemeier lattices

Let  $X$  be a Kählerian K3 surface  $X$  with a marking  $\tau : S \subset N_i$  by a Niemeier lattice  $N_i$ . By the conditions on  $\tau$  and our considerations in Sec. 3 using [12, Remark 1.14.7], we obtain the following result.

**Theorem 7.** *Let  $X$  be a Kählerian K3 surface with marking  $\tau : S \subset N_i$  by a Niemeier lattice  $N_i$ .*

*Then  $P(X) \cap S = P(S)$  is the basis for the root system  $\Delta(S)$  and  $P(S) = P(X) \cap S$  is a subset of  $P(N_i)$ . Thus, the Dynkin diagram  $\Gamma(P(X) \cap S)$  gives the Dynkin diagram of  $\Gamma(P(S))$ , and it is a subdiagram of the Dynkin diagram  $\Gamma(P(N_i))$ .*

*In particular, if  $S_X$  is negative definite, then  $S = S_X$ , and  $P(X) = P(N_i) \cap S$  gives the set of classes of all non-singular rational curves on  $X$ .*

*If  $X$  is algebraic, then  $P(X) \cap S = P(S) = P(N_i) \cap S$  gives the set of classes of all non-singular rational curves on  $X$  which are contracted by the linear system  $|nh|$  for  $n > 0$  where  $h$  is the primitive nef element in  $S_X$  which generates the orthogonal complement to  $S$  in  $S_X$ .*

This shows that for Kählerian K3 surfaces  $X$ , marking by Niemeier lattices describes the sets of non-singular rational curves on  $X$  such that their classes are contained in  $S \subset S_X$ .

We recall that an automorphism  $\phi$  of a Kählerian K3 surface  $X$  is called *symplectic* if  $\phi$  preserves the holomorphic form  $\omega_X$ , that is  $\phi^*(\omega_X) = \omega_X$ . This is equivalent (see [11]) that  $\phi$  gives the identity on the transcendental part

$$H^2(X, \mathbb{Z})/S_X. \quad (15)$$

It is known (see [14] and [2]; formally, it follows from the Global Torelli Theorem for K3 surfaces) that the kernel of the action of  $\text{Aut } X$  in  $H^2(X, \mathbb{Z})$  is trivial. A subgroup of  $\text{Aut } X$  is called *symplectic* if all its elements are symplectic. We denote by  $(\text{Aut } X)_0$  the group of all symplectic automorphisms of  $X$ .

For a marking  $\tau : S \subset N_i$  of  $X$  by a Niemeier lattice  $N_i$ , we shall consider the group of automorphisms

$$\text{Aut}(X, S)_0 = \{f \in (\text{Aut } X)_0 \mid f(S) = S \text{ and } f|_{S_{H^2(X, \mathbb{Z})}^\perp} \text{ is identity}\}. \quad (16)$$

If  $S_X$  is negative definite or hyperbolic, then, in (16), the additional condition that  $f|S_{H^2(X,\mathbb{Z})}^\perp$  is identity follows from other conditions. In all cases, the definition of  $\text{Aut}(X, S)_0$  is equivalent to

$$\text{Aut}(X, S)_0 = \{f \in \text{Aut } X \mid f(S) = S \text{ and } f|S_{H^2(X,\mathbb{Z})}^\perp \text{ is identity}\}. \quad (17)$$

By [12, Proposition 1.5.1],  $\phi \in O(S)$  can be extended to  $O(H^2(X, \mathbb{Z}))$  identically on  $S_{H^2(X,\mathbb{Z})}^\perp$  if and only if  $\phi$  gives identity on  $A_S = S^*/S$ . By the Global Torelli Theorem for K3 surfaces (see [14] and [2]), then  $\phi = f^*$  for some  $f \in \text{Aut } X$ . By (15),  $f \in (\text{Aut } X)_0$ . Further we identify  $\text{Aut}(X, S)_0$  with its action in  $S$ . Then, by our considerations in Sect. 3 which follow from [12, Remark 1.14.7], we obtain the following result.

**Theorem 8.** *Let  $X$  be a Kählerian K3 surface and  $S \subset N_i$  is its marking by a Niemeier lattice  $N_i$ ,  $i = 1, 2, \dots, 24$ .*

*Let  $\text{Aut}(X, S)_0$  be the symplectic automorphism group of  $X$  which consists of all automorphisms of  $X$  which give identity on  $S_{H^2(X,\mathbb{Z})}^\perp$  (all of them are symplectic). Then the action of  $\text{Aut}(X, S)_0$  in  $S$  identifies it with the subgroup  $A(S) \subset A(N_i)$  which is :*

$$A(S) = \{\phi \in A(N_i) \mid \phi|S_{N_i}^\perp \text{ is identity}\}. \quad (18)$$

We have

$$\text{Aut}(X, S)_0 = A(S)|S.$$

The Leech type sublattice  $\mathcal{L}(S) \subset S$  (or the coinvariant sublattice) is

$$\mathcal{L}(S) = S_{A(S)} = (S^{A(S)})^\perp_S = ((N_i)^{A(S)})^\perp_{N_i}. \quad (19)$$

Let  $P(X) \cap S$  be the set of all classes of non-singular rational curves of  $X$  which are contained in  $S$ . Then  $P(X) \cap S$  is invariant with respect to  $A(S)$ .

**Remark 9.** By these theorems, to construct Kählerian K3 surfaces with all possible markings  $S \subset N_i$  (further, we shall skip  $\tau$ ) and with all possible  $P(X) \cap S$  and  $\text{Aut}(X, S)_0$ , one can follow the following steps.

(a) For a Niemeier lattice  $N_i$  take a subgroup  $A \subset A(N_i)$  such that  $\mathcal{L} = (N_i)_A = ((N_i)^A)^\perp_{N_i}$  satisfies Theorem 3 (equivalently, there exists a primitive embedding  $\mathcal{L} \subset L_{K3}$ ). Further, we call such subgroups *KahK3 subgroups (that is Kählerian K3 surfaces subgroups)*. Let  $\text{Clos}(A) \subset A(N_i)$  (we use notation from [4]) is the maximal subgroup of  $A(N_i)$  with the same coinvariant lattice  $\mathcal{L} = (N_i)_A = ((N_i)^A)^\perp_{N_i}$ . These and only these subgroups  $\text{Clos}(A)$  correspond to the full symplectic automorphism groups  $\text{Aut}(X, S)_0$  of Kählerian K3 surfaces which can be marked by  $N_i$ . The subgroup  $A$  can be only a proper subgroup  $A \subset \text{Clos}(A) = \text{Aut}(X, S)_0$ .

We note that the list of all possible abstract such groups  $A$  for all Niemeier lattices together is known (see [11] for Abelian groups  $A$  and Mukai [7], Xiao [19] and Kondō [5] for arbitrary groups  $A$ ). The corresponding to  $A$  Leech type lattices  $\mathcal{L}$  (for all Niemeier lattices together) are also known (see [11] for Abelian groups  $A$  and Hashimoto [4] for arbitrary groups  $A$ ). Almost for all abstract groups  $A$ , the Leech type lattices  $\mathcal{L}$  with action of  $A$  on  $\mathcal{L}$  are uniquely defined up to isomorphisms.

(b) Choose a subset  $P \subset P(N_i)$  which is invariant with respect to  $A$  and such that the primitive sublattice  $S_0 = [\mathcal{L}, P]_{pr} \subset N_i$  generated by  $\mathcal{L}$  and  $P$  satisfies Theorem 3 (equivalently, there exists a primitive embedding  $S_0 = [\mathcal{L}, P]_{pr} \subset L_{K3}$ ). Then  $P \subset P(S_0) = S_0 \cap P(N_i)$  and  $A \subset A(S_0)$ .

(c) Extend the primitive sublattice above to a primitive sublattice  $S_0 \subset S \subset N_i$  such that  $S \cap P(N_i) = P$ ,

$$\{\phi \in A(N_i) \mid \phi(S) = S \text{ and } \phi|_{S_{N_i}^\perp} \text{ is identity}\} = Clos(A),$$

and  $S$  satisfies Theorem 3. Then  $A(S) = Clos(A)$ ,  $\mathcal{L}(S) = \mathcal{L}$ ,  $P(S) = P$ , and there exists a primitive embedding  $S \subset L_{K3}$ .

(d) Follow Remark 6 to construct a Kählerian K3 surface  $X$  with the marking  $S \subset N_i$  by the Niemeier lattice  $N_i$ . Then

$$\text{Aut}(X, S)_0 = Clos(A), \quad P(X) \cap S = P(S) = P.$$

## 6 Markings of K3 surfaces by some concrete Niemeier lattices ( $N_1$ — $N_{18}$ and $N_{23}$ )

Below, we use the basis of a root lattice  $A_n$ ,  $D_n$  or  $E_k$ ,  $k = 6, 7, 8$ , which is shown on Figure 1.

For  $A_n$ ,  $n \geq 1$ , we denote  $\epsilon_1 = (\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)/(n+1)$ . It gives the generator of the discriminant group  $A_n^*/A_n \cong \mathbb{Z}/(n+1)\mathbb{Z}$ .

For  $D_n$ ,  $n \geq 4$  and  $n \equiv 0 \pmod{2}$ , we denote  $\epsilon_1 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_{n-1})/2$ ,  $\epsilon_2 = (\alpha_{n-1} + \alpha_n)/2$ ,  $\epsilon_3 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_n)/2$ . They give all non-zero elements of the discriminant group  $D_n^*/D_n \cong (\mathbb{Z}/2\mathbb{Z})^2$ .

For  $D_n$ ,  $n \geq 4$  and  $n \equiv 1 \pmod{2}$ , we denote  $\epsilon_1 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-2})/2 + \alpha_{n-1}/4 - \alpha_n/4$ ,  $\epsilon_2 = (\alpha_{n-1} + \alpha_n)/2$ ,  $\epsilon_3 = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-2})/2 - \alpha_{n-1}/4 + \alpha_n/4$ . They give all non-zero elements of  $D_n^*/D_n \cong \mathbb{Z}/4\mathbb{Z}$ .

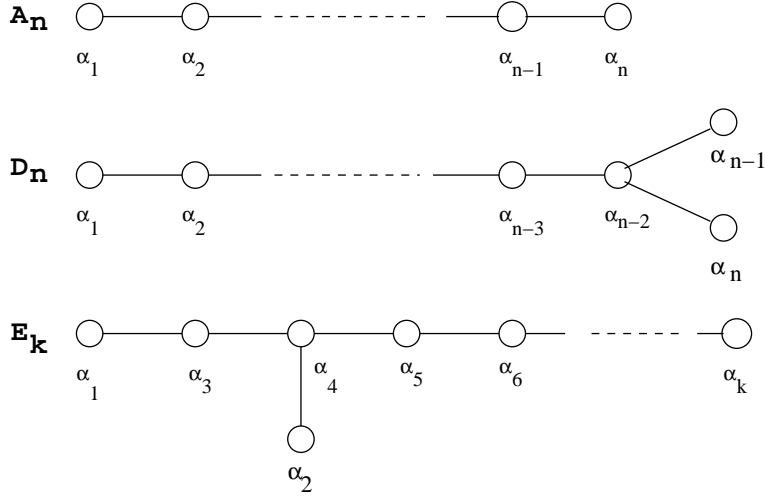


Figure 1: Bases of Dynkin diagrams  $A_n$ ,  $D_n$ ,  $E_k$ .

For  $E_6$ , we denote  $\varepsilon_1 = (\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)/3$ ,  $\varepsilon_2 = (-\alpha_1 + \alpha_3 - \alpha_5 + \alpha_6)/3$ . They give all non-zero elements of  $E_6^*/E_6 \cong \mathbb{Z}/3\mathbb{Z}$ .

For  $E_7$ , we denote  $\varepsilon_1 = (\alpha_2 + \alpha_5 + \alpha_7)/2$ . It gives the non-zero element of  $E_7^*/E_7 \cong \mathbb{Z}/2\mathbb{Z}$ .

If the Dynkin diagram of a root lattice has several connected components, the second index of a basis numerates the corresponding connected component.

Below we apply the results above to concrete cases of Niemeier lattices  $N_i$ . We denote by  $X$  a Kählerian K3 surface.

**Case 0.** Let us consider marking of  $X$  by negative definite even unimodular lattices  $K$  of the rank 16. There are 2 such lattices:  $K_1 = \Gamma_{16}$  with the root sublattice  $D_{16}$  and  $K_2 = 2E_8$  with the root sublattice  $2E_8$ . For this case, any primitive sublattice  $S \subset K_i$ ,  $i = 1, 2$ , satisfies Theorem 3 because  $\text{rk } S + l(A_S) \leq 16 < 22$  by Theorem 1. It gives marking of some  $X$  if  $P(S) \subset P(K_i)$ . It is why this case is easy. (Basically, this case was considered in [12, Remark 1.14.7].)

In this case,  $A(K_1)$  is trivial and  $A(K_2) = C_2$  is the group of order 2 which permutes two components  $E_8$ .

Let  $X$  be marked by a primitive sublattice  $S \subset K_1 = \Gamma_{16}$ . We have  $\text{Aut}(X, S)_0 = A(K_1)$  is trivial. We have  $P(X) \cap S = P(S)$  and  $\Gamma(P(S)) \subset \Gamma(P(K_1)) = \mathbb{D}_{16}$ . Any such subgraph is possible.

Let  $A = A(K_2)$  is the group of order 2. Then the coinvariant sublattice (or Leech type



sublattice) is

$$\mathcal{L}_A = (K_2)_A = E_8(2) = [\{\alpha_{i1} - \alpha_{i2} \mid 1 \leq i \leq 8\}] \subset K_2 = 2E_8.$$

It is isomorphic to  $E_8$  with the form multiplied by 2.

Let  $X$  be marked by  $S \subset K_2$ . Then  $P(X) \cap S = P(S)$ . If  $E_8(2) \not\subset S$ , then the symplectic group  $A(X, S)_0$  is trivial and  $\Gamma(P(X) \cap S) \subset 2\mathbb{E}_8$ . If  $E_8(2) \subset S$ , then  $A(X, S)_0 = A(K_2)$  is the group of order 2. The graph  $\Gamma(P(X) \cap S)$  is a subgraph of  $2\mathbb{E}_8$  which is invariant with respect to the permutation of two components of  $2\mathbb{E}_8$ . All such subgraphs are possible.

Further, firstly, we consider Niemeier lattices  $N_i$  such that  $A(N_i)$  has no non-trivial Kählerian K3 surfaces subgroups (equivalently, of KahK3 subgroups). Thus, if  $X$  has marking  $S \subset N_i$  by such Niemeier lattice  $N_i$ , then  $\text{Aut}(X, S)_0$  is trivial.

**Case 1.** The Niemeier lattice  $N_1 = N(D_{24}) = [D_{24}, \epsilon_1]$  has the trivial group  $A(N_1)$  (see [3, Ch. 16]).

Let  $X$  be marked by a primitive sublattice  $S \subset N_1$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_1)) = \mathbb{D}_{24}$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . The group  $\text{Aut}(X, S)_0 = A(N_1)$  is trivial.

Let  $S = D_n$ ,  $17 \leq n \leq 19$ . It satisfies Theorem 3. Thus, there exists  $X$  with marking  $S = D_n \subset N_1$ . By classification of Niemeier lattices, only  $N_1 = N(D_{24})$  is possible for marking of  $X$  with such lattice  $S \subset S_X$ .

**Case 2.** The Niemeier lattice  $N_2 = N(D_{16} \oplus E_8) = [D_{16}, \epsilon_{11}] \oplus E_8$  has the trivial group  $A(N_2)$  (see [3, Ch. 16]).

Let  $X$  be marked by a primitive sublattice  $S \subset N_2$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_2)) = \mathbb{D}_{16}\mathbb{E}_8$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . The group  $\text{Aut}(X, S)_0 = A(N_2)$  is trivial.

Let  $S = D_n \oplus E_8$ ,  $8 \leq n \leq 11$ . It satisfies Theorem 3. Thus, there exists  $X$  with marking  $S \subset N_2$ . By classification of Niemeier lattices, only  $N_2 = N(D_{16} \oplus E_8)$  is possible for marking of  $X$  with such  $S \subset S_X$ .

**Case 4.** The Niemeier lattice  $N = N_4 = N(A_{24}) = [A_{24}, 5\epsilon_1]$  has the group  $A(N)$  of the order 2 (see [3, Ch. 16]). It gives the non-trivial involution of the diagram  $P(N_4) = \mathbb{A}_{24}$ . It is easy to see that  $\mathcal{L} = N_{A(N)} = (N^{A(N)})^\perp_N$  does not satisfy Theorem 3. Thus, only a trivial subgroup  $A \subset A(N)$  is KahK3 subgroup. (It also follows from results of [11] where it is shown that  $\text{rk } \mathcal{L} = 8$  for KahK3 subgroup of order 2.)

Let  $X$  be marked by a primitive sublattice  $S \subset N_4$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_4)) = \mathbb{A}_{24}$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . The group  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = A_n$ ,  $18 \leq n \leq 19$ . It satisfies Theorem 3. Thus, there exists  $X$  with marking  $S \subset N_4 = N(A_{24})$ . By classification of Niemeier lattices, only  $N_1 = N(D_{24})$  and  $N_4 = N(A_{24})$  are possible for marking of  $X$  with such  $S \subset S_X$ .

**Case 5.** The Niemeier lattice  $N = N_5 = N(2D_{12}) = [2D_{12}, \varepsilon_{11} + \varepsilon_{22}, \varepsilon_{21} + \varepsilon_{12}]$  has  $A(N)$  of order 2 which permutes two components  $D_{12}$ . It is easy to see that  $\mathcal{L} = N_{A(N)} = (N^{A(N)})^\perp_N$  does not satisfy Theorem 3. Thus, only a trivial subgroup  $A \subset A(N)$  is KahK3 subgroup.

Let  $X$  be marked by a primitive sublattice  $S \subset N_5$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_5)) = 2\mathbb{D}_{12}$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . The group  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = D_{10} \oplus D_9$ . It satisfies Theorem 3. Thus, there exists  $X$  with marking  $S \subset N_5$ . By classification of Niemeier lattices, only  $N_5 = N(2D_{12})$  is possible for marking of  $X$  with such  $S \subset S_X$ .

**Case 10.** The Niemeier lattice  $N = N_{10} = N(2A_{12}) = [2A_{12}, \varepsilon_{11} + 5\varepsilon_{12}]$  has  $A(N)$  which is a cyclic group of order 4. It is easy to see that  $\mathcal{L} = N_A = (N^A)^\perp_N$  does not satisfy Theorem 3 for both its non-trivial subgroups  $A \subset A(N)$ . It is enough to check this for its subgroup of the order 2 which gives non-trivial involutions on the graph  $\mathbb{A}_{12}$  of each of two components  $A_{12}$ . Thus, only a trivial subgroup  $A \subset A(N)$  is KahK3 subgroup.

Let  $X$  be marked by a primitive sublattice  $S \subset N_{10}$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{10})) = 2\mathbb{A}_{12}$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . The group  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = A_{10} \oplus A_9$ . It satisfies Theorem 3. Thus, there exists  $X$  with marking  $S \subset N_{10}$ . By classification of Niemeier lattices, only  $N_1 = N(D_{24})$ ,  $N_4 = N(A_{24})$ ,  $N_5 = N(2D_{12})$  and  $N_{10} = N(2A_{12})$  are possible for marking of  $X$  with such  $S \subset S_X$ .

For next cases, the group  $A(N_i)$  has Kählerian K3 surfaces subgroups (that is KahK3 subgroups)  $A$  only of the order 1 or 2. Thus, if  $X$  is marked by one of these lattices then the group  $\text{Aut}(X, S)_0$  is either trivial, or it has the order 2.

**Case 3.** For the Niemeier lattice  $N = N_3 = N(3E_8)$ , the group  $A(N)$  is  $\mathfrak{S}_3$  (obviously) which acts by permutations on the three components  $E_8$  which we denote by  $(E_8)_j$ ,  $j = 1, 2, 3$ . Simple calculations show that for  $A \subset A(N) = \mathfrak{S}_3$  the lattice  $\mathcal{L} = N_A = ((N^A)^\perp_N)$  does not satisfy Theorem 3 if  $A$  is  $\mathfrak{S}_3$  or the alternating group  $\mathfrak{A}_3$ . It is valid only if  $A$  is either generated by a transposition, or  $A$  is trivial. They give all KahK3 subgroups  $A \subset A(N)$ .

For  $A = [(kl)] \subset A(N)$  generated by a transposition  $(kl)$ ,  $1 \leq k < l \leq 3$ , the Leech type (or the coinvariant) sublattice is

$$E_8(2)_{kl} = N_A = [\{\alpha_{ik} - \alpha_{il} \mid 1 \leq i \leq 8\}].$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_3 = N(3E_8)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_3)) = 3\mathbb{E}_8$ . Any such  $S$  gives marking of some  $X$  and

$P(X) \cap S = P(S)$ . If  $S$  does not contain each of sublattices  $E_8(2)_{kl}$ ,  $1 \leq k < l \leq 3$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2)_{kl} \subset S$  for some  $1 \leq k < l \leq 3$ , then  $\text{Aut}(X, S)_0 = [(kl)]$  is the group of order 2.

Let  $S = 2E_8$ . Then  $S$  satisfies Theorem 3 and gives marking  $S \subset N_3 = N(3E_8)$  of some  $X$ . By classification of Niemeier lattices, only  $N_3 = N(3E_8)$  is possible for marking of  $X$  with such  $S \subset S_X$ .

**Case 6.** For the Niemeier lattice  $N = N_6 = N(A_{17} \oplus E_7) = [A_{17} \oplus E_7, 3\varepsilon_{11} + \varepsilon_{12}]$ , the group  $A(N_6)$  has order 2 (see [3, Ch. 16]). Its generator is trivial on  $E_7$  and gives the non-trivial involution of the diagram  $\mathbb{A}_{17}$ .

The coinvariant sublattice for  $A(N)$  is equal to

$$E_8(2) = N_{A(N)} = [\{\alpha_{i1} - \alpha_{(18-i)1} \mid 1 \leq i \leq 8\}, \frac{1}{3} \sum_{i=1}^8 i(\alpha_{i1} - \alpha_{(18-i)1})] \subset N_6.$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_6 = N(A_{17} \oplus E_7)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_6)) = \mathbb{A}_{17}\mathbb{E}_7$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . If  $E_8(2) \not\subset S$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2) \subset S$ , then  $\text{Aut}(X, S)_0 = A(N_6)$  has order two.

In particular,  $S = [A_{17}, 6\varepsilon_{11}] \subset N_6$  satisfies Theorem 3 and gives marking of some  $X$ . By classification of Niemeier lattices,  $X$  can be marked by  $N_6 = N(A_{17} \oplus E_7)$  only for such  $S \subset S_X$ .

**Case 7.** For the Niemeier lattice  $N = N_7 = N(D_{10} \oplus 2E_7) = [D_{10} \oplus 2E_7, \varepsilon_{11} + \varepsilon_{12}, \varepsilon_{31} + \varepsilon_{13}]$ , the group  $A(N_7)$  has order 2 (see [3, Ch. 16]). Its generator permutes two diagrams  $\mathbb{E}_7$  and gives a non-trivial involution on the diagram  $\mathbb{D}_{10}$ .

For  $A(N)$ , the coinvariant sublattice is equal to

$$E_8(2) = N_{A(N)} = [\alpha_{91} - \alpha_{10,1}, \{\alpha_{i2} - \alpha_{i3} \mid 1 \leq i \leq 7\},$$

$$(\alpha_{91} - \alpha_{10,1} + \alpha_{22} - \alpha_{23} + \alpha_{52} - \alpha_{53} + \alpha_{72} - \alpha_{73})/2] \subset N_7.$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_7 = N(D_{10} \oplus 2E_7)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_7)) = \mathbb{D}_{10}2\mathbb{E}_7$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . If  $E_8(2) \not\subset S$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2) \subset S$ , then  $\text{Aut}(X, S)_0 = A(N_7)$  has order two.

In particular,  $S = [\alpha_{91}, \alpha_{10,1}, 2E_7]_{pr} \subset N_7$  satisfies Theorem 3 and gives marking of some  $X$ . By classification of Niemeier lattices, then  $X$  can be marked by  $N_7 = N(D_{10} \oplus 2E_7)$  only for such  $S \subset S_X$ .

**Case 8.** For the Niemeier lattice  $N = N_8 = N(A_{15} \oplus D_9) = [A_{15} \oplus D_9, 2\varepsilon_{11} + \varepsilon_{12}]$ , the group  $A(N_8)$  has order 2 (see [3, Ch. 16]). Its generator gives a non-trivial involution on the diagrams  $\mathbb{A}_{15}$  and  $\mathbb{D}_9$ .

For  $A(N)$ , the coinvariant sublattice is equal to

$$E_8(2) = N_A = [\{\alpha_{i1} - \alpha_{(16-i)1} \mid 1 \leq i \leq 7\}, \alpha_{82} - \alpha_{92},$$

$$\frac{1}{4} \sum_{i=1}^7 (\alpha_{i1} - \alpha_{(16-i)1}) + \frac{1}{2}(\alpha_{82} - \alpha_{92})] \subset N_8.$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_8 = N(A_{15} \oplus D_9)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_8)) = \mathbb{A}_{15}\mathbb{D}_9$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . If  $E_8(2) \not\subset S$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2) \subset S$ , then  $\text{Aut}(X, S)_0 = A(N_8)$  has order two.

In particular,  $S = [A_{15}, \alpha_{82}, \alpha_{92}]_{pr} \subset N_8$  satisfies Theorem 3 and gives marking of some  $X$ . By classification of Niemeier lattices, it is easy to see that  $X$  can be marked by  $N_8 = N(A_{15} \oplus D_9)$  only for such  $S \subset S_X$ .

**Case 9.** For the Niemeier lattice

$$N = N_9 = N(3D_8) = [3D_8, \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{12} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{22} + \varepsilon_{13}]$$

the group  $A(N)$  is  $\mathfrak{S}_3$  which acts by permutations on the three components  $D_8$  and by permutations of  $\alpha_{71}$ ,  $\alpha_{72}$  and  $\alpha_{73}$ . Simple calculations show that for  $A \subset A(N) = \mathfrak{S}_3$  the lattice  $\mathcal{L} = N_A = ((N^A)_N^\perp)$  does not satisfy Theorem 3 if  $A$  is  $\mathfrak{S}_3$  or  $\mathfrak{A}_3$ . It is valid only if  $A$  is either generated by a transposition, or  $A$  is trivial. They give all KahK3 subgroups.

Let  $A = [(kl)]$ ,  $1 \leq k < l \leq 3$ , is a transposition on  $(D_8)_j$ ,  $j = 1, 2, 3$ . Then the coinvariant sublattice is

$$N_A = E_8(2)_{kl} =$$

$$[\{\alpha_{ik} - \alpha_{il} \mid 1 \leq i \leq 8\}, (\alpha_{1k} - \alpha_{1l} + \alpha_{3k} - \alpha_{3l} + \alpha_{5k} - \alpha_{5l} + \alpha_{8k} - \alpha_{8l})/2].$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_9 = N(3D_8)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_9)) = 3\mathbb{D}_8$ . Any such  $S$  gives marking of some  $X$  and  $P(S) = P(X) \cap S$ . If  $S$  does not contain each of sublattices  $E_8(2)_{kl}$ ,  $1 \leq k < l \leq 3$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2)_{kl} \subset S$  for some  $1 \leq k < l \leq 3$ , then  $\text{Aut}(X, S)_0 = [(kl)]$  is the group of order 2.

Let  $S = [2D_8]_{pr} \subset N_9$ . Then  $S$  satisfies Theorem 3 and gives marking of some  $X$ , moreover,  $\text{Aut}(X, S)_0$  has order 2. By classification of Niemeier lattices,  $X$  can be marked by the Niemeier lattice  $N_9 = N(3D_8)$  only for such  $S \subset S_X$ .

**Case 11.** For the Niemeier lattice  $N = N_{11} = N(A_{11} \oplus D_7 \oplus E_6) = [A_{11} \oplus D_7 \oplus E_6, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13}]$ , the group  $A(N_{11})$  has order 2 (see [3, Ch. 16]). Its generator gives a non-trivial involution on the subdiagrams  $\mathbb{A}_{11}$ ,  $\mathbb{D}_7$  and  $\mathbb{E}_6$ . For the group  $A = A(N_{11})$ , the coinvariant sublattice is equal to

$$E_8(2) = N_A = [\{\alpha_{i1} - \alpha_{(12-i)1} \mid 1 \leq i \leq 5\}, \alpha_{62} - \alpha_{72}, \alpha_{13} - \alpha_{63}, \alpha_{33} - \alpha_{53},$$

$$\frac{1}{6} \sum_{i=1}^5 i(\alpha_{i1} - \alpha_{(12-i)1}) + \frac{1}{2}(\alpha_{62} - \alpha_{72}) + \frac{1}{3}(-\alpha_{13} + \alpha_{63} + \alpha_{33} - \alpha_{53})] \subset N_{11}.$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_{11} = N(A_{11} \oplus D_7 \oplus E_6)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{11})) = \mathbb{A}_{11}\mathbb{D}_7\mathbb{E}_6$ . Any such  $S$  gives marking of some  $X$  and  $P(S) = P(X) \cap S$ . If  $S$  does not contain  $E_8(2)$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2) \subset S$ , then  $\text{Aut}(X, S)_0 = A(N_{11})$  is the group of order 2.

In particular,  $S = [A_{11} \oplus E_6, \alpha_{62}, \alpha_{72}]_{pr} \subset N_{11}$  satisfies Theorem 3 and gives marking of some  $X$ . By classification of Niemeier lattices, it is easy to see that  $X$  can be marked by  $N_{11} = N(A_{11} \oplus D_7 \oplus E_6)$  only for such  $S \subset S_X$ .

**Case 15.** For the Niemeier lattice

$$N = N_{15} = N(3A_8) = [3A_8, 4\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{23}, \varepsilon_{21} + 4\varepsilon_{12} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{22} + 4\varepsilon_{13}]$$

the group  $A(N)$  has the order 12, and it is the direct product of the group of order 2 which gives non-trivial involutions on all three components  $\mathbb{A}_8$ , and the group  $\mathfrak{S}_3$  which acts by permutations of the three components  $\mathbb{A}_8$  and of  $\alpha_{11}$ ,  $\alpha_{12}$  and  $\alpha_{13}$  (see [3, Ch. 16]).

Simple calculations show that for  $A \subset A(N)$  the coinvariant sublattice  $\mathcal{L} = N_A = ((N^A)^\perp_N)$  satisfies Theorem 3 only if  $A$  is either trivial or  $A = [(kl)]$ ,  $1 \leq k < l \leq 3$ , is generated by the transposition  $(kl)$  of two components  $(\mathbb{A}_8)_k$  and  $(\mathbb{A}_8)_l$  and  $\alpha_{1k}$ ,  $\alpha_{1l}$ , and it is identity on the remaining component  $(\mathbb{A}_8)_j$ . They give all KahK3 subgroups. For  $A = [(kl)]$ , the coinvariant sublattice is

$$E_8(2)_{kl} = N_A = [\{\alpha_{ik} - \alpha_{il} \mid 1 \leq i \leq 8\}, \frac{1}{3} \sum_{i=1}^8 i(\alpha_{ik} - \alpha_{il})] \subset N_{15}.$$

Let  $X$  be marked by a primitive sublattice  $S \subset N_{15} = N(3A_8)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{15})) = 3\mathbb{A}_8$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ . If  $S$  does not contain each of sublattices  $E_8(2)_{kl}$ ,  $1 \leq k < l \leq 3$ , then  $\text{Aut}(X, S)_0$  is trivial. If  $E_8(2)_{kl} \subset S$  for some  $1 \leq k < l \leq 3$ , then  $\text{Aut}(X, S)_0 = [(kl)]$  is the group of order 2.

Let  $S = [2A_8]_{pr} \subset N_{15}$ . Then  $S$  satisfies Theorem 3 and gives marking of some  $X$ , moreover,  $\text{Aut}(X, S)_0$  has order 2. By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{15} = N(3A_8)$  only for such  $S \subset S_X$ .

**Some general remarks.** Cases below are more complicated. We use the following simple general statements and computer Programs (see Appendix, Sect. 7). (Of course, one can also use them for all cases which we considered above and skipped calculations because they were easy.)

**Proposition 10.** *Let  $N$  be a Niemeier (or any other even negative definite unimodular lattice),  $P(N)$  a basis of the root system of  $N$  and*

$$A(N) = \{\phi \in O(N) \mid \phi(P(N)) = P(N)\}.$$

*If  $A \subset A(N)$  is a KahK3 subgroup with the Leech type (or coinvariant) sublattice  $N_A = (N^A)^\perp_N$  (equivalently, there exists a primitive embedding  $N_A \subset L_{K3}$ ), then its conjugate  $A^g = gAg^{-1}$ ,  $g \in A(N)$ , is also KahK3 subgroup with the Leech type (or coinvariant sublattice)  $N_{A^g} = g(N_A)$ . They define KahK3 conjugacy classes of  $A(N)$ .*

*Thus, to describe KahK3 subgroups  $A \subset A(N)$  and their coinvariant sublattices, it is enough to describe representatives of all KahK3 conjugacy classes in  $A(N)$  and their coinvariant sublattices.*

*Proof.* Indeed,  $N_{A^g}$  is isomorphic to  $N_A$ . Therefore, if  $N_A$  has a primitive embedding  $N_A \subset L_{K3}$ , then  $N_{A^g} = g(N_A) \subset N$  also has a primitive embedding  $N_{A^g} \subset L_{K3}$ .  $\square$

To calculate the coinvariant sublattices, we can use

**Proposition 11.** *Let  $N$  be a Niemeier (or any other even negative definite unimodular lattice),  $P(N)$  a basis of the root system of  $N$  and*

$$A(N) = \{\phi \in O(N) \mid \phi(P(N)) = P(N)\}.$$

*Let  $A_1 \subset A(N)$  and  $A_2 \subset A(N)$  are two subgroups, and  $A = \langle A_1, A_2 \rangle \subset A(N)$  is generated by  $A_1$  and  $A_2$ . Then the coinvariant sublattice  $N_A$  is the primitive sublattice  $N_A = [N_{A_1}, N_{A_2}]_{pr} \subset N$  of  $N$  generated by the coinvariant sublattices  $N_{A_1}$  and  $N_{A_2}$  of  $A_1$  and  $A_2$ . Therefore,  $A = \langle A_1, A_2 \rangle$  is KahK3 subgroup if and only if  $[N_{A_1}, N_{A_2}]_{pr}$  has a primitive embedding into  $L_{K3}$ .*

*In particular, if  $A = \langle g_1, \dots, g_n \rangle$  is generated by  $g_1, \dots, g_n \in A$ , then  $N_A = [N_{\langle g_1 \rangle}, \dots, N_{\langle g_n \rangle}]_{pr} \subset N$ . Moreover,  $A$  is a KahK3 subgroup if and only if the sublattice  $[N_{\langle g_1 \rangle}, \dots, N_{\langle g_n \rangle}]_{pr} \subset N$  has a primitive embedding into  $L_{K3}$ .*

*Proof.* Indeed, since  $A_1, A_2 \subset A$ , then  $[N_{A_1}, N_{A_2}]_{pr} \subset N_A$ . Let  $M = [N_{A_1}, N_{A_2}]_{pr} \subset N$ . Since  $N_{A_1} \subset M$  and  $N_{A_2} \subset M$ , it follows that  $A_1 \subset A(M)$ ,  $A_2 \subset A(M)$ , and  $A = \langle A_1, A_2 \rangle \subset A(M)$ . It follows that  $A = \langle A_1, A_2 \rangle$  gives identity on  $M_N^\perp$ , and  $M \subset N_{\langle A_1, A_2 \rangle}$ . Therefore,  $M = N_A$ .  $\square$

To calculate the coinvariant sublattice  $N_A$  for a subgroup  $A \subset A(N)$ , we also use the following important and simple statement.

**Proposition 12.** *Let  $N$  be a Niemeier (or any other even unimodular lattice),  $P(N)$  a basis of the root system of  $N$  and*

$$A(N) = \{\phi \in O(N) \mid \phi(P(N)) = P(N)\}.$$

*Suppose that  $P(N)$  generates  $N$  over  $\mathbb{Q}$  (otherwise, use an  $A$ -invariant basis of  $N \otimes \mathbb{Q}$  instead of  $P(N)$ ). Let  $P(N) = \{e_1, e_2, \dots, e_n\}$  and  $\{e_1^*, e_2^*, \dots, e_n^*\} \subset N \otimes \mathbb{Q}$  are dual elements, that is  $e_i^* \cdot e_j = \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker symbol.*

*Let  $A \subset A(N)$  be a subgroup and  $N_A \subset N$  its coinvariant sublattice  $N_A = (N^A)^\perp_N$ . Let*

$$\{e_{i_{11}}, \dots, e_{i_{1l_1}}\}, \dots, \{e_{i_{t1}}, \dots, e_{i_{tl_t}}\}$$

*be all orbits of  $A$  in  $P(N)$ , and  $t$  the number of orbits.*

*Then*

$$\{e_{i_{11}}^* - e_{i_{12}}^*, \dots, e_{i_{1(l_1-1)}}^* - e_{i_{1l_1}}^*\}, \dots, \{e_{i_{t1}}^* - e_{i_{t2}}^*, \dots, e_{i_{t(l_t-1)}}^* - e_{i_{tl_t}}^*\}$$

*give basis of  $N_A \otimes \mathbb{Q}$ . In particular,  $\text{rk } N_A = \text{rk } N - t$ .*

A primitive sublattice  $K_{pr} \subset N$  of a lattice  $N$  is defined by its rational basis for the vector space  $K \otimes \mathbb{Q}$ . Really,  $K_{pr} = K \otimes \mathbb{Q} \cap N \subset N \otimes \mathbb{Q}$ . Thus, Proposition 12 gives an efficient method for calculation of  $N_A$  and its invariants (for Theorem 3) to find out if  $N_A$  has a primitive embedding into  $L_{K3}$  and  $A$  is KahK3 subgroup.

In Appendix (Sect. 7) we give a Program 0 (which uses Programs 1 – 4) which calculates a normal (or Smith) basis of the primitive sublattice  $K_{pr} \subset N$  for any sublattice  $K$  of  $N$  which is given by generators of  $K \otimes \mathbb{Q}$ . We use the root basis  $P(N)$  of a Niemeier lattice  $N$  (corresponding to basic columns  $(0, \dots, 0, 1, 0, \dots, 0)^t$  of the length 24), the integer matrix  $r$  of  $N$  in this basis (which is equivalent to Dynkin diagram), and additional cording data of  $N$ . The sublattice  $K$  is given by rational columns in this basis (matrix SUBL of size  $(24 \times \cdot)$  in Program 0. The Program 0 calculates the normal (elementary divisors or Smith) basis of  $K_{pr}$  (denoted by SUBLpr for the Program 0) for the embedding  $K_{pr} \subset K_{pr}^*$ . It calculates elementary divisors (or Smith) invariants of this embedding (denoted by DSUBLpr for the

Program 0). In particular, it calculates  $\text{rk } K_{pr}$ , the discriminant group  $A_{K_{pr}} = K_{pr}^*/K_{pr}$ , gives the number  $l(A_{K_{pr}})$  of its minimal generators. Moreover, the normal basis can be used to calculate the Jordan form of  $K_{pr} \otimes \mathbb{Z}_p$  over the rings of  $p$ -adic integers, and the discriminant form  $q_{K_{pr}}$ . The last vectors of this basis give the unimodular part of  $K_{pr}$  over  $\mathbb{Z}_p$ . Thus, Program 0 (Appendix, Sect. 7) gives all necessary invariants for Theorem 3 to find out if  $K_{pr}$  has a primitive embedding into  $L_{K_3}$ . The program is very fast for all cases which we consider below.

**Case 13.** For the Niemeier lattice

$$N = N_{13} = N(2A_9 \oplus D_6) = [2A_9 \oplus D_6, 2\varepsilon_{11} + 4\varepsilon_{12}, 5\varepsilon_{11} + \varepsilon_{13}, 5\varepsilon_{12} + \varepsilon_{33}],$$

the group  $A(N)$  is a cyclic group  $C_4$  of the order 4 (see [3, Ch. 16]). Its elements are defined by permutations of the terminals of Dynkin diagrams  $\mathbb{A}_9$  and  $\mathbb{D}_6$ . Its generator  $\varphi$  gives the permutation

$$\varphi = \begin{pmatrix} \alpha_{11} & \alpha_{91} & \alpha_{12} & \alpha_{92} & \alpha_{53} & \alpha_{63} \\ \alpha_{12} & \alpha_{92} & \alpha_{91} & \alpha_{11} & \alpha_{63} & \alpha_{53} \end{pmatrix}.$$

By Proposition 12, the coinvariant sublattice  $N_{[\varphi]}$  is

$$N_{[\varphi]} = [\alpha_{11}^* - \alpha_{12}^*, \alpha_{12}^* - \alpha_{91}^*, \alpha_{91}^* - \alpha_{92}^*, \alpha_{21}^* - \alpha_{22}^*, \alpha_{22}^* - \alpha_{81}^*, \alpha_{81}^* - \alpha_{82}^*, \alpha_{31}^* - \alpha_{32}^*, \alpha_{32}^* - \alpha_{71}^*, \\ \alpha_{71}^* - \alpha_{72}^*, \alpha_{41}^* - \alpha_{42}^*, \alpha_{42}^* - \alpha_{61}^*, \alpha_{61}^* - \alpha_{62}^*, \alpha_{51}^* - \alpha_{52}^*, \alpha_{53}^* - \alpha_{63}^*]_{pr} \subset N.$$

Using Program 0 (see Appendix, Sect. 7), we obtain that  $\text{rk } N_{[\varphi]} = 14$ , and  $N_{[\varphi]}^*/N_{[\varphi]} \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$ , and  $l(N_{[\varphi]}^*/N_{[\varphi]}) = 6$ . By Theorem 3, the lattice  $N_{[\varphi]}$  has a primitive embedding into  $L_{K_3}$ , and  $[\varphi] = A(N)$  is KahK3 group.

Any subgroup of  $A(N)$  is also KahK3 subgroup. The only non-trivial subgroup of  $A(N)$  is  $[\varphi^2]$  of order 2. Similar calculations show that  $N_{[\varphi^2]} \cong E_8(2)$ .

Let  $X$  be marked by a primitive sublattice  $S \subset N_{13} = N(2A_9 \oplus D_6)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{13})) = 2\mathbb{A}_9 \mathbb{D}_6$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ .

If  $N_{[\varphi]} \subset S$ , then  $\text{Aut}(X, S)_0 = [\varphi] \cong C_4$  is cyclic group of the order 4. Otherwise, if only  $N_{[\varphi^2]} \subset S$ , then  $\text{Aut}(X, S)_0 = [\varphi^2] \cong C_2$ . Otherwise, if  $N_{[\varphi^2]} \not\subset S$ , then  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = [2A_9, N_{[\varphi]}]_{pr} = [\alpha_{11}, \alpha_{21}, \dots, \alpha_{81}, N_{[\varphi]}]_{pr} \subset N_{13}$ . Then (using Program 0 in Appendix, Sect. 7) we have  $\text{rk } S = 19$  and  $S^*/S \cong \mathbb{Z}/4\mathbb{Z}$ . Thus  $S$  satisfies Theorem 3 and gives marking of some  $X$ . We have  $\text{Aut}(X, S)_0 \cong C_4$  and  $2\mathbb{A}_9 \subset \Gamma(P(X) \cap S) = P(S)$ . By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{13} = N(2A_9 \oplus D_6)$  only for such  $S \subset S_X$ .



**Case 16.** For the Niemeier lattice

$$N = N_{16} = N(2A_7 \oplus 2D_5) = [2A_7 \oplus 2D_5, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{13} + \varepsilon_{24}, \varepsilon_{11} + 7\varepsilon_{12} + \varepsilon_{23} + \varepsilon_{14}],$$

the group  $A(N)$  is the dihedral group  $\mathfrak{D}_8$  of the order 8 (see [3, Ch. 16]). We identify it with the group of symmetries of a quadrat. Its elements are defined by permutations of the terminals of Dynkin diagrams  $\mathbb{A}_7$  and  $\mathbb{D}_5$ . The central symmetry  $\varphi_0$  gives the involution

$$\varphi_0 = \begin{pmatrix} \alpha_{11} & \alpha_{71} & \alpha_{12} & \alpha_{72} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \\ \alpha_{71} & \alpha_{11} & \alpha_{72} & \alpha_{12} & \alpha_{53} & \alpha_{43} & \alpha_{54} & \alpha_{44} \end{pmatrix}.$$

Two generating symmetries  $\varphi_1$  and  $\varphi_2$  give the involutions

$$\varphi_1 = \begin{pmatrix} \alpha_{11} & \alpha_{71} & \alpha_{12} & \alpha_{72} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \\ \alpha_{12} & \alpha_{72} & \alpha_{11} & \alpha_{71} & \alpha_{43} & \alpha_{53} & \alpha_{54} & \alpha_{44} \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} \alpha_{11} & \alpha_{71} & \alpha_{12} & \alpha_{72} & \alpha_{43} & \alpha_{53} & \alpha_{44} & \alpha_{54} \\ \alpha_{11} & \alpha_{71} & \alpha_{72} & \alpha_{12} & \alpha_{44} & \alpha_{54} & \alpha_{43} & \alpha_{53} \end{pmatrix}$$

where  $\varphi_2\varphi_1$  gives a rotation by  $90^\circ$ , and has order 4.

For the cyclic subgroup  $H = [\varphi_2\varphi_1]$  of order 4, the coinvariant sublattice  $N_H$  has  $\text{rk } N_H = 16$  and  $l(N_H^*/N_H) = 8$  (we use Proposition 12 and Program 0 of Appendix, Sect. 7). Thus,  $N_H$  has no primitive embeddings into  $L_{K3}$ , and  $H$  is not KahK3 subgroup by Theorem 3. It follows that  $A(N)$  is not KahK3 subgroup either.

For the conjugate in  $A(N)$  subgroups  $H = [\varphi_0, \varphi_1]$  and  $H = [\varphi_0, \varphi_2]$  isomorphic to  $C_2 \times C_2$ , we have  $\text{rk } N_H = 12$  and  $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$  (we use Proposition 12 and Program 0 of Appendix, Sect. 7). Thus,  $N_H$  has a primitive embedding into  $L_{K3}$  by Theorem 3, and the  $H \cong C_2 \times C_2$  are KahK3 subgroups.<sup>1</sup>

All other non-trivial subgroups of  $A(N)$  are cyclic subgroups of order 2 which are  $[\varphi_0]$ , and conjugate subgroups  $[\varphi_1]$ ,  $[\varphi_1\varphi_0]$ ,  $[\varphi_2]$ ,  $[\varphi_2\varphi_0]$  of order 2. They are KahK3 subgroups since they are subgroups of  $H$  above. The coinvariant sublattices  $N_{[\varphi_0]}$ ,  $N_{[\varphi_1]}$ ,  $N_{[\varphi_1\varphi_0]}$ ,  $N_{[\varphi_2]}$ ,  $N_{[\varphi_2\varphi_0]}$  are isomorphic to  $E_8(2)$ .

Let  $X$  be marked by a primitive sublattice  $S \subset N_{16} = N(2A_7 \oplus 2D_5)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{16})) = 2\mathbb{A}_7 2\mathbb{D}_5$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ .

If  $N_{[\varphi_0, \varphi_1]} \subset S$ , then  $\text{Aut}(X, S)_0 = [\varphi_0, \varphi_1] \cong C_2 \times C_2$ . If  $N_{[\varphi_0, \varphi_2]} \subset S$ , then  $\text{Aut}(X, S)_0 = [\varphi_0, \varphi_2] \cong C_2 \times C_2$ . Otherwise, if only  $N_{[\varphi]} \subset S$  for one of  $\varphi \in \{\varphi_0, \varphi_1, \varphi_1\varphi_0, \varphi_2, \varphi_2\varphi_0\}$ ,

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<sup>1</sup>These calculations show that for a symplectic group  $G = C_2 \times C_2$  on a Kählerian K3 surface, the group  $S_{(G)}^*/S_{(G)} = S_{(2,2)}^*/S_{(2,2)} \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$ . We must correct our calculation of this group in [11, Prop. 10.1].

then  $\text{Aut}(X, S)_0 = [\varphi] \cong C_2$ . Otherwise, if  $N_{[\varphi]} \not\subset S$  for each such  $\varphi$ , then  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = [2A_7, N_{[\varphi_0, \varphi_1]}]_{pr} \subset N_{16}$ . Then (using Program 0 in Appendix, Sect. 7) we have  $\text{rk } S = 16$  and  $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^4$ . Thus  $S$  satisfies Theorem 3 and gives marking of some  $X$ . We have  $\text{Aut}(X, S)_0 \cong C_2 \times C_2$  and  $2\mathbb{A}_7 \subset \Gamma(P(X) \cap S) = P(S)$ . By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{16} = N(2A_7 \oplus 2D_5)$  only for such  $S \subset S_X$ .

*Below, in our calculations, we don't mention using of Proposition 12 and Program 0 of Appendix, Sec. 7. We use them all the time.*

**Case 17.** For the Niemeier lattice

$$N = N_{17} = N(4A_6) =$$

$$= [4A_6, \varepsilon_{11} + 2\varepsilon_{12} + \varepsilon_{13} + 6\varepsilon_{14}, \varepsilon_{11} + 6\varepsilon_{12} + 2\varepsilon_{13} + \varepsilon_{14}, \varepsilon_{11} + \varepsilon_{12} + 6\varepsilon_{13} + 2\varepsilon_{14}],$$

the group  $A(N)$  has the center  $[\varphi_0]$  of order 2 which preserves components  $4\mathbb{A}_6$ , and  $A(N)/[\varphi_0] = \mathfrak{A}_4$  is the alternating group on components of  $4\mathbb{A}_6$  (see [3, Ch. 16]). Its elements are defined by permutations of the terminals of the Dynkin diagrams  $\mathbb{A}_6$ . It is generated by the involution

$$\varphi_0 = \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{61} & \alpha_{11} & \alpha_{62} & \alpha_{12} & \alpha_{63} & \alpha_{13} & \alpha_{64} & \alpha_{14} \end{pmatrix},$$

and by elements  $\varphi_1$  and  $\varphi_4$  of order 3,

$$\varphi_1 = \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{11} & \alpha_{61} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} & \alpha_{12} & \alpha_{62} \end{pmatrix},$$

$$\varphi_4 = \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{12} & \alpha_{62} & \alpha_{63} & \alpha_{13} & \alpha_{61} & \alpha_{11} & \alpha_{14} & \alpha_{64} \end{pmatrix}$$

where  $\varphi_4\varphi_1$  has order 4 and  $(\varphi_4\varphi_1)^2 = \varphi_0$ . All elements of  $A(N)$  are conjugate to  $\varphi_0$ ,  $\varphi_1$ ,  $\varphi_0\varphi_1$  (of order 6) and  $\varphi_4\varphi_1$ .

The coinvariant sublattice  $N_{[\varphi_0]}$  has  $\text{rk } N_{[\varphi_0]} = 12$  and  $N_{[\varphi_0]}^*/N_{[\varphi_0]} \cong (\mathbb{Z}/2\mathbb{Z})^{12}$ . By Theorem 3,  $N_{[\varphi_0]}$  has no primitive embeddings into  $L_{K3}$ , and  $[\varphi_0]$  is not KahK3 subgroup.

For the cyclic group  $[\varphi_1] \cong C_3$ , we have  $\text{rk } N_{[\varphi_1]} = 12$  and  $N_{[\varphi_1]}^*/N_{[\varphi_1]} \cong (\mathbb{Z}/3\mathbb{Z})^6$ . By Theorem 3,  $N_{[\varphi_1]}$  has a primitive embedding into  $L_{K3}$ , and  $[\varphi_1]$  is KahK3 subgroup. Its conjugate give all cyclic subgroups of  $A(N)$  of order 3. They are  $[\varphi_1]$ ,  $[\varphi_4]$  and  $[\varphi_2]$ ,  $[\varphi_3]$  where

$$\varphi_2 = \varphi_1\varphi_4\varphi_1^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{13} & \alpha_{63} & \alpha_{12} & \alpha_{62} & \alpha_{64} & \alpha_{14} & \alpha_{61} & \alpha_{11} \end{pmatrix},$$

$$\varphi_3 = \varphi_1^2 \varphi_4 \varphi_1^{-2} = \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{14} & \alpha_{64} & \alpha_{61} & \alpha_{11} & \alpha_{13} & \alpha_{63} & \alpha_{62} & \alpha_{12} \end{pmatrix},$$

From the structure of  $A(N)$ , it follows that *conjugate subgroups*

$$H = [\varphi_i] \cong C_3, \quad i = 1, 2, 3, 4,$$

are all non-trivial KahK3 subgroups of  $A(N_{17})$ . We have  $\text{rk } N_H = 12$ ,  $N_H^*/N_H \cong (\mathbb{Z}/3\mathbb{Z})^6$ .

Let  $X$  be marked by a primitive sublattice  $S \subset N_{17} = N(4A_6)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{17})) = 4A_6$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ .

If  $N_{[\varphi_i]} \subset S$ , for one of  $i = 1, 2, 3, 4$ , then  $\text{Aut}(X, S)_0 = [\varphi_i] \cong C_3$ . Otherwise, if  $N_{[\varphi_i]} \not\subset S$  for all  $i = 1, 2, 3, 4$ , then  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = [(A_6)_2 = [\alpha_{12}, \dots, \alpha_{62}], N_{[\varphi_1]}]_{pr} \subset N_{17}$ . We have  $\text{rk } S = 18$ ,  $S^*/S \cong \mathbb{Z}/7\mathbb{Z}$ . Thus,  $S$  satisfies Theorem 3 and gives marking of some  $X$ . We have  $\text{Aut}(X, S)_0 \cong C_3$  and  $3A_6 \subset \Gamma(P(X) \cap S) = P(S)$ . By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{17} = N(4A_6)$  only for such  $S \subset S_X$ .

**Case 14.** For the Niemeier lattice

$$N = N_{14} = N(4D_6) = [4D_6, \text{ even permutations of } 0_1 + \varepsilon_{12} + \varepsilon_{23} + \varepsilon_{34}] = [4D_6, \varepsilon_{12} + \varepsilon_{23} + \varepsilon_{34}, \\ \varepsilon_{32} + \varepsilon_{13} + \varepsilon_{24}, \varepsilon_{22} + \varepsilon_{33} + \varepsilon_{14}; \varepsilon_{11} + \varepsilon_{33} + \varepsilon_{24}, \varepsilon_{21} + \varepsilon_{13} + \varepsilon_{34}, \varepsilon_{31} + \varepsilon_{23} + \varepsilon_{14}; \varepsilon_{21} + \varepsilon_{32} + \varepsilon_{14}, \\ \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{34}, \varepsilon_{31} + \varepsilon_{12} + \varepsilon_{24}; \varepsilon_{31} + \varepsilon_{22} + \varepsilon_{13}, \varepsilon_{11} + \varepsilon_{32} + \varepsilon_{23}, \varepsilon_{21} + \varepsilon_{12} + \varepsilon_{33}]$$

the group  $A(N)$  can be identified with  $\mathfrak{S}_4$  by its action on components of  $4\mathbb{D}_6$  (see [3, Ch. 16]). Its elements are defined by permutations of the terminals of the connected components of Dynkin diagrams  $4\mathbb{D}_6$ . We numerate these components by 1, 2, 3, 4. Even permutations give the corresponding permutations of  $(\alpha_{51}, \alpha_{52}, \alpha_{53}, \alpha_{54})$  and  $(\alpha_{61}, \alpha_{62}, \alpha_{63}, \alpha_{64})$ . Odd permutations change  $\alpha_{5i}$  and  $\alpha_{6i}$  places. For example, the transposition (12) gives the permutation

$$(12) = \begin{pmatrix} \alpha_{51} & \alpha_{61} & \alpha_{52} & \alpha_{62} & \alpha_{53} & \alpha_{63} & \alpha_{54} & \alpha_{64} \\ \alpha_{62} & \alpha_{52} & \alpha_{61} & \alpha_{51} & \alpha_{63} & \alpha_{53} & \alpha_{64} & \alpha_{54} \end{pmatrix}.$$

For  $(12)(34) \in \mathfrak{S}_4$  of order 2, we have  $\text{rk } N_{[(12)(34)]} = 12$  and  $N_{[(12)(34)]}^*/N_{[(12)(34)]} \cong (\mathbb{Z}/2\mathbb{Z})^{12}$ . By Theorem 3,  $N_{[(12)(34)]}$  has no primitive embeddings into  $L_{K3}$ , and  $[(12)(34)]$  and its conjugate are not KahK3 subgroups.

Let  $(\mathfrak{D}_6)_4 = \mathfrak{S}_3(1, 2, 3)$  consists of all elements of  $\mathfrak{S}_4$  which preserve 4. Its conjugate we similarly denote as  $(\mathfrak{D}_6)_k$ ,  $k = 1, 2, 3, 4$ . They are isomorphic to the dihedral group  $\mathfrak{D}_6$  of order 6. We have

$$\text{rk } N_{(\mathfrak{D}_6)_4} = 14, \quad N_{(\mathfrak{D}_6)_4}^*/N_{(\mathfrak{D}_6)_4} \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3.$$

By Theorem 3,  $N_{(\mathfrak{D}_6)_4}$  has a primitive embedding into  $L_{K3}$ , and  $(\mathfrak{D}_6)_4$  is KahK3 subgroup.

Subgroups of  $(\mathfrak{D}_6)_4$  are also KahK3 subgroups. Non-trivial ones are conjugate to  $[(12)] \cong C_2$  and  $(C_3)_4 = [(123)] \cong C_3$ . We have  $N_{[(12)]} \cong E_8(2)$ , and

$$\text{rk } N_{[(123)]} = 12, \quad N_{[(123)]}^*/N_{[(123)]} \cong (\mathbb{Z}/3\mathbb{Z})^6.$$

From the structure of  $\mathfrak{S}_4$ , we obtain that *all non-trivial KahK3 subgroups of  $A(N_{14})$  are conjugate subgroups  $(\mathfrak{D}_6)_k$ ,  $k = 1, 2, 3, 4$ , isomorphic to  $\mathfrak{D}_6$ ; conjugate subgroups  $[(ij)]$ ,  $1 \leq i < j \leq 4$ , isomorphic to  $C_2$ ; conjugate subgroups  $(C_3)_k \subset (\mathfrak{D}_6)_k$ ,  $k = 1, 2, 3, 4$ , isomorphic to  $C_3$ .*

Let  $X$  be marked by a primitive sublattice  $S \subset N_{14} = N(4D_6)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{14})) = 4\mathbb{D}_6$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ .

If  $N_{(\mathfrak{D}_6)_k} \subset S$  for one of  $k = 1, 2, 3, 4$ , then  $\text{Aut}(X, S)_0 = (\mathfrak{D}_6)_k \cong \mathfrak{D}_6$ . Otherwise, if  $N_{(C_3)_k} \subset S$  for one of  $k = 1, 2, 3, 4$ , only, then  $\text{Aut}(X, S)_0 = (C_3)_k \cong C_3$ . Otherwise, if  $N_{[(ij)]} \subset S$  for  $1 \leq i < j \leq 4$ , only, then  $\text{Aut}(X, S)_0 = [(ij)] \cong C_2$ . Otherwise,  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = [(D_6)_1] = [\alpha_{11}, \dots, \alpha_{61}]$ ,  $N_{(\mathfrak{D}_6)_4}]_{pr} \subset N_{14}$ . We have  $\text{rk } S = 19$ ,  $S^*/S \cong \mathbb{Z}/4\mathbb{Z}$ . Thus,  $S$  satisfies Theorem 3 and gives marking of some  $X$ . We have  $\text{Aut}(X, S)_0 \cong \mathfrak{D}_6$  and  $3\mathbb{D}_6 \subset \Gamma(P(X) \cap S) = P(S)$ . By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{14} = N(4D_6)$  only for such  $S \subset S_X$ .

**Case 12.** For the Niemeier lattice

$$N = N_{12} = N(4E_6) = [4E_6, \varepsilon_{11} + \varepsilon_{13} + \varepsilon_{24}, \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{14}, \varepsilon_{11} + \varepsilon_{12} + \varepsilon_{23}],$$

the group  $A(N)$  has the center  $[\varphi_0]$  of order 2 which preserves components  $4\mathbb{E}_6$ , and  $A(N)/[\varphi_0] = \mathfrak{S}_4$  is the symmetric group on components of  $4\mathbb{E}_6$  (see [3, Ch. 16]). Its elements are defined by permutations of the terminals of the Dynkin diagrams  $\mathbb{E}_6$ . It is generated by the involutions

$$\begin{aligned} \varphi_0 &= \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{61} & \alpha_{11} & \alpha_{62} & \alpha_{12} & \alpha_{63} & \alpha_{13} & \alpha_{64} & \alpha_{14} \end{pmatrix}, \\ \widetilde{(12)} &= \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{12} & \alpha_{62} & \alpha_{11} & \alpha_{61} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \end{pmatrix}, \\ \widetilde{(23)} &= \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{61} & \alpha_{11} & \alpha_{13} & \alpha_{63} & \alpha_{12} & \alpha_{62} & \alpha_{14} & \alpha_{64} \end{pmatrix}, \\ \widetilde{(34)} &= \begin{pmatrix} \alpha_{11} & \alpha_{61} & \alpha_{12} & \alpha_{62} & \alpha_{13} & \alpha_{63} & \alpha_{14} & \alpha_{64} \\ \alpha_{61} & \alpha_{11} & \alpha_{12} & \alpha_{62} & \alpha_{14} & \alpha_{64} & \alpha_{13} & \alpha_{63} \end{pmatrix}. \end{aligned}$$

Similarly we define involutions  $\widetilde{(ij)}$ ,  $1 \leq i < j \leq 4$ , which act as transpositions of components  $(\mathbb{E}_6)_i$ ,  $(\mathbb{E}_6)_j$  and elements  $\alpha_{1i}$ ,  $\alpha_{1j}$ .

The element  $\widetilde{(34)}\widetilde{(12)}$  has order 4, and  $(\widetilde{(34)}\widetilde{(12)})^2 = \varphi_0$ . The coinvariant sublattice  $N_{[\widetilde{(34)}\widetilde{(12)}]}$  has  $\text{rk } N_{[\widetilde{(34)}\widetilde{(12)}]} = 16$ , and  $N_{[\widetilde{(34)}\widetilde{(12)}]}^*/N_{[\widetilde{(34)}\widetilde{(12)}]} = (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^4$ . By Theorem 3,  $N_{[\widetilde{(34)}\widetilde{(12)}]}$  has no primitive embeddings into  $L_{K3}$ , and  $[\widetilde{(34)}\widetilde{(12)}]$  is not KahK3 subgroup.

For  $k = 1, 2, 3, 4$ , let  $(\mathfrak{D}_{12})_k$  consists of all elements of  $A(N_{12})$  which preserve the component  $(\mathbb{E}_6)_k$ . The conjugate subgroups  $(\mathfrak{D}_{12})_k$  are isomorphic to the dihedral group  $\mathfrak{D}_{12}$  of order 12. We have

$$\text{rk } N_{(\mathfrak{D}_{12})_k} = 16, \quad N_{(\mathfrak{D}_{12})_k}^*/N_{(\mathfrak{D}_{12})_k} \cong (\mathbb{Z}/6\mathbb{Z})^4.$$

By Theorem 3,  $N_{(\mathfrak{D}_{12})_k}$  have primitive embeddings into  $L_{K3}$ , and  $(\mathfrak{D}_{12})_k$  are KahK3 subgroups.

From the structure of  $A(N)$ , it follows that *conjugate subgroups*

$$(\mathfrak{D}_{12})_k \cong \mathfrak{D}_{12}, \quad k = 1, 2, 3, 4,$$

and all their subgroups give all KahK3 subgroups of  $A(N_{12})$ .

These non-trivial subgroups are (where  $k$  marks subgroups of  $(\mathfrak{D}_{12})_k$ ):

isomorphic to  $C_6$  conjugate to  $(C_6)_4 = [\widetilde{(12)}\widetilde{(23)}]$  subgroups

$$(C_6)_k, \quad k = 1, 2, 3, 4,$$

with  $N_{(C_6)_k} = N_{(\mathfrak{D}_{12})_k}$  (therefore,  $\text{Clos}(C_6)_k = (\mathfrak{D}_{12})_k$ ); isomorphic to  $\mathfrak{D}_6$  subgroups

$$(\mathfrak{D}_6)_{ki}, \quad k = 1, 2, 3, 4, \quad i = 1, 2,$$

with  $\text{rk } N_{(\mathfrak{D}_6)_{ki}} = 14$ , and  $N_{(\mathfrak{D}_6)_{ki}}^*/N_{(\mathfrak{D}_6)_{ki}} \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3$ ; isomorphic to  $C_3$  conjugate subgroups

$$(C_3)_k, \quad k = 1, 2, 3, 4,$$

with  $\text{rk } N_{(C_3)_k} = 12$  and  $N_{(C_3)_k}^*/N_{(C_3)_k} \cong (\mathbb{Z}/3\mathbb{Z})^6$ ; isomorphic to  $C_2 \times C_2$  conjugate subgroups

$$[\widetilde{(ij)}, \varphi_0], \quad 1 \leq i < j \leq 4,$$

with  $N_{[\widetilde{(ij)}, \varphi_0]} = 12$ , and  $N_{[\widetilde{(ij)}, \varphi_0]}^*/N_{[\widetilde{(ij)}, \varphi_0]} \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$ ; isomorphic to  $C_2$  subgroups

$$[\varphi_0], \quad [\widetilde{(ij)}], \quad [\widetilde{(ij)}\varphi_0], \quad 1 \leq i < j \leq 4$$

with coinvariant lattices isomorphic to  $E_8(2)$ .

Let  $X$  be marked by a primitive sublattice  $S \subset N_{12} = N(4E_6)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{17})) = 4\mathbb{E}_6$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ .

If  $N_{(\mathfrak{D}_{12})_k} \subset S$  for one of  $k = 1, 2, 3, 4$ , then  $\text{Aut}(X, S)_0 = (\mathfrak{D}_{12})_k \cong \mathfrak{D}_{12}$ . Otherwise, if  $N_{(\mathfrak{D}_6)_{ki}} \subset S$  only for one of  $k = 1, 2, 3, 4$ ,  $i = 1, 2$ , then  $\text{Aut}(X, S)_0 = (\mathfrak{D}_6)_{ki} \cong \mathfrak{D}_6$ . Otherwise, if  $N_{(C_3)_k} \subset S$  only for one of  $k = 1, 2, 3, 4$ , then  $\text{Aut}(X, S)_0 = (C_3)_k \cong C_3$ . Otherwise, if  $N_{[(ij), \varphi_0]} \subset S$  only for one of  $1 \leq i < j \leq 4$ , then  $\text{Aut}(X, S)_0 = [(ij), \varphi_0] \cong C_2 \times C_2$ . Otherwise, if  $N_H \subset S$  only for one of  $H \in \{[\varphi_0]\} \cup \{[(ij)], [(ij)\varphi_0], | 1 \leq i < j \leq 4\}$ , then  $\text{Aut}(X, S)_0 = H \cong C_2$ . Otherwise,  $\text{Aut}(X, S)_0$  is trivial.

Let  $S = [(E_6)_1 = [\alpha_{11}, \dots, \alpha_{61}], N_{(C_3)_4}]_{pr} \subset N_{12}$ . We have  $\text{rk } S = 18$ ,  $S^*/S \cong \mathbb{Z}/3\mathbb{Z}$ . Thus,  $S$  satisfies Theorem 3 and gives marking of some  $X$ . We have  $\text{Aut}(X, S)_0 \cong C_3$  and  $3\mathbb{E}_6 \subset \Gamma(P(X) \cap S) = P(S)$ . By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{12} = N(4E_6)$  only for such  $S \subset S_X$ .

**Case 18.** For the Niemeier lattice

$$N = N_{18} = N(4A_5 \oplus D_4) = [4A_5 \oplus D_4, 2\varepsilon_{11} + 2\varepsilon_{13} + 4\varepsilon_{14}, 2\varepsilon_{11} + 4\varepsilon_{12} + 2\varepsilon_{14},$$

$$2\varepsilon_{11} + 2\varepsilon_{12} + 4\varepsilon_{13}, 3\varepsilon_{11} + 3\varepsilon_{12} + \varepsilon_{15}, 3\varepsilon_{11} + 3\varepsilon_{13} + \varepsilon_{25}, 3\varepsilon_{11} + 3\varepsilon_{14} + \varepsilon_{35}]$$

the group  $A = A(N)$  has the center  $[\varphi_0]$  of order 2 which preserves components  $4\mathbb{A}_5$  and  $\mathbb{D}_4$ . The group  $A(N)/[\varphi_0] = \mathfrak{S}_4$  is the symmetric group on components of  $4\mathbb{A}_5$  (see [3, Ch. 16]). Elements of  $A$  are defined by their actions on the terminals of the Dynkin diagrams  $4\mathbb{A}_5\mathbb{D}_4$ . It is generated by the involutions

$$\begin{aligned} \varphi_0 &= \begin{pmatrix} \alpha_{11} & \alpha_{51} & \alpha_{12} & \alpha_{52} & \alpha_{13} & \alpha_{53} & \alpha_{14} & \alpha_{54} & \alpha_{15} & \alpha_{35} & \alpha_{45} \\ \alpha_{51} & \alpha_{11} & \alpha_{52} & \alpha_{12} & \alpha_{53} & \alpha_{13} & \alpha_{54} & \alpha_{14} & \alpha_{55} & \alpha_{35} & \alpha_{45} \end{pmatrix}, \\ \widetilde{(12)} &= \begin{pmatrix} \alpha_{11} & \alpha_{51} & \alpha_{12} & \alpha_{52} & \alpha_{13} & \alpha_{53} & \alpha_{14} & \alpha_{54} & \alpha_{15} & \alpha_{35} & \alpha_{45} \\ \alpha_{12} & \alpha_{52} & \alpha_{11} & \alpha_{51} & \alpha_{13} & \alpha_{53} & \alpha_{14} & \alpha_{54} & \alpha_{15} & \alpha_{35} & \alpha_{45} \end{pmatrix}, \\ \widetilde{(23)} &= \begin{pmatrix} \alpha_{11} & \alpha_{51} & \alpha_{12} & \alpha_{52} & \alpha_{13} & \alpha_{53} & \alpha_{14} & \alpha_{54} & \alpha_{15} & \alpha_{35} & \alpha_{45} \\ \alpha_{51} & \alpha_{11} & \alpha_{13} & \alpha_{53} & \alpha_{12} & \alpha_{52} & \alpha_{14} & \alpha_{54} & \alpha_{45} & \alpha_{35} & \alpha_{15} \end{pmatrix}, \\ \widetilde{(34)} &= \begin{pmatrix} \alpha_{11} & \alpha_{51} & \alpha_{12} & \alpha_{52} & \alpha_{13} & \alpha_{53} & \alpha_{14} & \alpha_{54} & \alpha_{15} & \alpha_{35} & \alpha_{45} \\ \alpha_{51} & \alpha_{11} & \alpha_{12} & \alpha_{52} & \alpha_{14} & \alpha_{54} & \alpha_{13} & \alpha_{53} & \alpha_{35} & \alpha_{15} & \alpha_{45} \end{pmatrix}. \end{aligned}$$

Similarly we define involutions  $\widetilde{(ij)}$ ,  $1 \leq i < j \leq 4$ , which act as transpositions of components  $(\mathbb{A}_5)_i$ ,  $(\mathbb{A}_5)_j$  and elements  $\alpha_{1i}$ ,  $\alpha_{1j}$ .

The group  $A = A(N)$  is isomorphic to the group  $T_{48} = Q_8 \rtimes \mathfrak{S}_3$  (standard notation, see [7]). The coinvariant sublattice  $N_A$  has  $\text{rk } N_A = 19$  and  $N_A^*/N_A \cong \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

By Theorem 3,  $N_A$  has a primitive embedding into  $L_{K3}$ , and  $A = A(N)$  is a KahK3 subgroup. Therefore, any its subgroup  $H \subset A$  is also a KahK3 subgroup. It seems, these facts were first observed in [4].

The group  $A = A(N)$  has, up to conjugation, the following and only the following non-trivial subgroups  $H \subset A$  and invariants of their coinvariant sublattices  $N_H$ .

(1) A subgroup  $H$  isomorphic to  $T_{24}$  (the binary tetrahedral group) consists of elements of  $N$  which give even permutations of four components of  $4\mathbb{A}_5$ . The coinvariant sublattice  $N_H = N_A$ . Therefore,  $\text{Clos}(H) = A$  is isomorphic to  $T_{48}$ , and  $\text{rk } N_H = 19$ ,  $N_H^*/N_H \cong \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

(2) Subgroups  $H$  isomorphic to  $SD_{16}$  (2-Sylow subgroups of  $A$ ) are conjugate to  $[(\widetilde{(34)(23)(12)}), (\widetilde{(13)}), (\widetilde{(24)})]$  with  $\text{rk } N_H = 18$  and  $N_H^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

(3) Subgroups  $H$  isomorphic to  $C_8$  are conjugate to  $[(\widetilde{(34)(23)(12)})]$ , with  $\text{Clos}([( \widetilde{(34)(23)(12)} )]) = [(\widetilde{(34)(23)(12)}), (\widetilde{(13)}), (\widetilde{(24)})]$  isomorphic to  $SD_{16}$ . Thus,  $N_{[(\widetilde{(34)(23)(12)})]} = N_{[(\widetilde{(34)(23)(12)}), (\widetilde{(13)}), (\widetilde{(24)})]}$  and  $\text{rk } N_H = 18$ ,  $N_H^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

(4) A subgroup  $H$  isomorphic to  $Q_8$  is  $H = [(\widetilde{(34)(12)}), (\widetilde{(24)(13)})]$  with  $\text{rk } N_H = 17$  and  $N_H^*/N_H \cong (\mathbb{Z}/8\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^3$ .

(5) Subgroups  $H$  isomorphic to  $\mathfrak{D}_{12}$  are conjugate to  $[(\widetilde{(12)}), (\widetilde{(23)})]$  with  $\text{rk } N_H = 16$ ,  $N_H^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4$ .

(6) Subgroups  $H$  isomorphic to  $C_6$  are conjugate to  $[(\widetilde{(23)(12)})]$  with  $\text{Clos}([( \widetilde{(23)(12)} )]) = [(\widetilde{(12)}), (\widetilde{(23)})]$  isomorphic to  $\mathfrak{D}_{12}$ . Thus,  $N_{[(\widetilde{(23)(12)})]} = N_{[(\widetilde{(12)}), (\widetilde{(23)})]}$  and  $\text{rk } N_H = 16$  and  $N_H^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^4$ .

(7) Subgroups  $H$  isomorphic to  $\mathfrak{D}_8$  are conjugate to  $[(\widetilde{(12)}), (\widetilde{(34)})]$  with  $\text{rk } N_H = 15$ ,  $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^5$ .

(8) Subgroups  $H$  isomorphic to  $\mathfrak{D}_6$  are conjugate to  $[\varphi_0(\widetilde{(23)}), (\widetilde{(12)})]$  or  $[\varphi_0(\widetilde{(13)}), (\widetilde{(23)})]$  with  $\text{rk } N_H = 14$  and  $N_H^*/N_H \cong (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^3$ .

(9) Subgroups  $H$  isomorphic to  $C_4$  are conjugate to  $[(\widetilde{(34)(12)})]$ , with  $\text{rk } N_H = 14$  and  $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$ .

(10) Subgroups  $H$  isomorphic to  $C_2 \times C_2$  are  $[(\widetilde{(ij)}), \varphi_0]$ ,  $1 \leq i < j \leq 4$ , with  $\text{rk } N_H = 12$  and  $N_H^*/N_H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^6$ .

(11) Subgroups  $H$  isomorphic to  $C_3$  are conjugate to  $[\varphi_0(\widetilde{(23)(12)})]$  with  $\text{rk } N_H = 12$  and  $N_H^*/N_H \cong (\mathbb{Z}/3\mathbb{Z})^6$ .

(12) Subgroups  $H$  isomorphic to  $C_2$  are  $[\varphi_0]$ ,  $[(\widetilde{(ij)})]$  and  $[(\widetilde{(ij)})\varphi_0]$ ,  $1 \leq i < j \leq 4$ . The lattice  $N_H \cong E_8(2)$ .

Let  $X$  be marked by a primitive sublattice  $S \subset N = N_{18} = N(4A_5 \oplus D_4)$ . Then  $S$  must satisfy Theorem 3 and  $\Gamma(P(S)) \subset \Gamma(P(N_{18})) = 4\mathbb{A}_5\mathbb{D}_4$ . Any such  $S$  gives marking of some  $X$  and  $P(X) \cap S = P(S)$ .

If  $S = N_A$ , then  $\text{Aut}(X, S)_0 = A \cong T_{48}$ . Otherwise, if only  $N_H \subset S$ ,  $H \cong SD_{16}$  or  $H \cong \mathfrak{D}_{12}$ , then  $\text{Aut}(X, S)_0 = H$ . Otherwise, if only  $N_H \subset S$ ,  $H \cong Q_8$ ,  $\mathfrak{D}_8$  or  $\mathfrak{D}_6$ , then  $\text{Aut}(X, S)_0 = H$ . Otherwise, if only  $N_H \subset S$ ,  $H \cong C_4$  or  $H \cong C_3$ , then  $\text{Aut}(X, S)_0 = H$ . Otherwise, if only  $N_H \subset S$ ,  $H \cong C_2$ , then  $\text{Aut}(X, S)_0 = H \cong C_2$ . Otherwise,  $\text{Aut}(X, S)_0$  is trivial.

Let  $H = [\widetilde{(12)}, \widetilde{(34)}] \cong \mathfrak{D}_8$ . Let  $S = [(A_5)_1 = [\alpha_{11}, \dots, \alpha_{51}], N_H]_{pr} \subset N_{18}$ . We have  $\text{rk } S = 18$ ,  $S^*/S \cong (\mathbb{Z}/4\mathbb{Z})^4 \times (\mathbb{Z}/2\mathbb{Z})^2$  and  $q_{S_2} = q_\theta^{(2)}(2) \oplus q'$ . Thus,  $S$  satisfies Theorem 3 and gives marking of some  $X$ . We have  $H \subset \text{Aut}(X, S)_0$ ,  $H \cong \mathfrak{D}_8$ , and  $2\mathbb{A}_5 \subset \Gamma(P(X) \cap S) = P(S)$ . By classification of Niemeier lattices and our calculations above,  $X$  can be marked by the Niemeier lattice  $N_{18} = N(4A_5 \oplus D_4)$  only for such  $S \subset S_X$ .

Below, we consider a case when  $\text{Aut}(X, S)_0$  can be very large.

**Case 23.** (This is related to [17].) Let us assume that a K3 surface  $X$  has 16 non-intersecting non-singular rational curves. It is known [9] that classes

$$\alpha_1, \alpha_2, \dots, \alpha_{16}$$

of these curves generate a primitive sublattice  $\Pi \subset S_X$  which was described in [14] and [9], and  $X$  is a Kummer surface.

Let us consider marking  $S \subset N_i$  of  $X$  by  $S$  which contains the primitive sublattice  $\Pi \subset S$ . It is known [9] that

$$P(X) \cap S = \{\alpha_1, \dots, \alpha_{16}\}.$$

Thus,  $16\mathbb{A}_1$  should be a subgraph of the Dynkin graph of  $N_i$ , and the corresponding sublattice  $\Pi \subset N_i$  is a primitive sublattice. By classification of Niemeier lattices, this is possible only for  $N(24A_1)$ . Thus,  $X$  can be marked by the Niemeier lattice  $N_{23} = N(24A_1)$  only. For this case,  $\text{Aut}(X, S)_0$  is a subgroup of

$$Kum = \{\phi \in A(N(24A_1)) \mid \phi(\Pi) = \Pi\}.$$

In [17], in particular, some of these subgroups were considered.

In particular, our considerations above give the following result.

**Theorem 13.** *For each of Niemeier lattices  $N_i$ ,  $i = 1, 2, 3, 5-9, 11-18, 23$ , there exists a Kählerian K3 surface  $X$  such that  $X$  can be marked by the Niemeier lattice  $N_i$  only.*



We believe that the same result is valid for the remaining Niemeier lattices  $N_4, N_{10}, N_{19}, N_{20}, N_{21}$  and  $N_{22}$ . By Kondō's trick from [5] which we mentioned in Sec. 4, any Kählerian K3 surface which is marked by the Leech lattice  $N_{24}$  can be also marked by one of Niemeier lattices  $N_i, i = 1 - 23$ . On the other hand, it is natural to mark K3 surfaces with empty set  $P(X) \cap S$  by the Leech lattice  $N_{24}$ . *All Niemeier lattices are important for marking of Kählerian K3 surfaces.*

We hope to consider in more details remaining cases  $N_k, 19 \leq k \leq 24$ , in further publications.

## 7 Appendix: Programs

Here we give programs using GP/PARI Calculator, Version 2.2.13.

Program 0: niemeier\generall.txt

```
\\for Niemeier lattice Niem given by
\\the 24\times 24 symmetric matrix r in root basis
\\represented by basis of the size 24 vectors (0,0,...,0,1,0,...,0)~
\\and its rational cording matrix
\\cord of size ( . \times 24)
\\and its sublattice SUBL given by
\\ (24\times . ) rational matrix SUBL
\\it calculates basis of its primitive sublattice
\\SUBLpr (SUBL\otimes Q)\cap Niem as matrix SUBLpr
\\and calculates invariants DSUBLpr of
\\SUBLpr\subset SUBLpr*
\\and calculates the matrix rSUBLpr of SUBLpr in this
\\elementary divisors (Smith) basis SUBLpr
a=matrix(24,24+matsize(cord)[1]);
for(i=1,24,a[i,i]=1);for(i=1,matsize(cord)[1],a[,24+i]=cord[i,]~ );
L=a;N=SUBL;
\r niemeier\latt4.txt;
SUBLpr1=Npr;R=r;B=SUBLpr1;
\r niemeier\latt2.txt;
SUBLpr=BB;DSUBLpr=D;rSUBLpr=G;
```

Program 1: niemeier\latt1.txt

```
\\for a non-degenerate lattice
```

```

\\L given by a symmetric integer matrix l
\\in some generators
\\calculates the elementary divisors (Smith) basis of L
\\as a matrix b and
\\calculates the matrix ll=b~*l*b
\\of L in the bases b
\\calculates invariants d of L\subset L*
ww=matsnf(l,1);uu=ww[1];vv=ww[2];dd=ww[3];
nn=matsize(l)[1];nnn=nn;for(i=1,nn,if(dd[i,i]==0,nnn=nnn-1));
b=matrix(nn,nnn,X,Y,vv[X,Y+nn-nnn]);
ll=b~*l*b;
d=vector(nnn,X,dd[X+nn-nnn,X+nn-nnn]);
kill(ww);kill(uu);kill(vv);kill(dd);kill(nn);kill(nnn);

```

Program 2: niemeier\latt2.txt

```

\\for a non-degenerate lattice L
\\given by an integer quadratic n x n matrix R
\\and generators B (n x m) matrix with rational coefficients
\\calculates invariants D of L\subset L* of this lattice
\\the elementary divisors (Smith) basis BB of this lattice, (n x mm) matrix,
\\and matrix G=BB~*R*BB of L in this basis
l=B~*R*B;
\r niemeier\latt1.txt;
BB=B*b;G=BB~*R*BB;D=d;

```

Program 3: niemeier\latt3.txt

```

\\for a module M
\\given by rational columns
\\of matrix M, it finds its basis
\\as a matrix MM
\\and finds matrix VV such that MM=M*VV
gg=gcd(M);M1=M/gg;
ww=matsnf(M1,1);uu=ww[1];vv=ww[2];dd=ww[3];
mm=matsize(dd)[1];nn=matsize(dd)[2];
nnn=nn;for(i=1,nn,if(dd[i,i]==0,nnn=nnn-1));
VV=matrix(nn,nnn);
nnnn=0;for(i=1,nn,if(dd[i,i]==0,,nnnn=nnnn+1;VV[,nnnn]=vv[,i]));

```

```

M2=M1*VV;MM=M2*gg;
kill(gg);kill(M1);kill(ww);kill(uu);kill(vv);kill(dd);kill(mm);
kill(nn);kill(nnn);kill(nnnn);kill(M2);

```

Program 4: niemeier\latt4.txt

```

\\For a module L given by
\\a rational m x n matrix L
\\such that L contains all basic columns
\\(0,..,0,1,0,..0)~
\\and its submodule N given by rational
\\matrix N
\\it finds basis of the primitive
\\submodule  $N_{pr}=(N \otimes Q) \cap L$ 
\\as the matrix Npr
ggg=gcd(N);N1=N/ggg;
M=L;
\r niemeier\latt3.txt;
L1=MM;kill(VV);
N2=L1^-1*N1;
ww=matsnf(N2,1);uu=ww[1];vv=ww[2];dd=ww[3];
N3=N2*vv;mm=matsize(dd)[1];nn=matsize(dd)[2];
nnn=nn;for(i=1,nn,if(dd[i]==0,nnn=nnn-1));
N4=matrix(mm,nnn);
nnnn=0;
for(i=1,nn,if(dd[i]==0,nnnn=nnnn+1;ddd=gcd(dd[i]);\
N4[,nnnn]=N3[,i]/ddd));
Npr=L1*N4;
kill(ggg);kill(N1);kill(M);kill(L1);kill(MM);
kill(N2);kill(ww);kill(uu);kill(vv);kill(dd);
kill(N3);kill(mm);kill(nn);kill(nnn);kill(nnnn);
kill(ddd);kill(N4);

```

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